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Complex Analysis for Engineering Mathematics to Calculate the Residue and Evaluation of Real Definite Integrals

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Abstract: **Complex Analysis is playing an important role in various engineering fields and in particular contour integration. In this paper, I started with some important basic concepts of Analytic function, Singularities, Zeros , Isolated Singularity, Removable Singularity, Pole, Essential Singularity, and Residue to evaluate the complex integration.**

Keywords: **Analytic function, Singularities, Zeros, Isolated Singularity, Removable Singularity, Pole, Essential Singularity, Residue and Contour Integration.**

1. INTRODUCTION

Definition1. (Analytic function)

A single valued function f (z) is said to be analytic in a region R of complex plane if f (z) has derivative at each point of R. If the function f (z) is said to be analytic at a point z = α then f (z) is analytic in the region which contain the point z = α as interior. Hence a function f (z) is analytic at a point z = α, then f (z) is analytic in some neighborhood $D_R(\alpha) = \{z: |z - \alpha| < R\}$ of α .

Example1.1. Any polynomial function (real or complex) is analytic

Example1.2. The exponential function is analytic

Example1.3. The trigonometric functions are analytic

Example1.4. The logarithmic functions are analytic

Example1.5. The absolute function |z| is not analytic, because it is not differentiable at z = 0

Singularities and Zeros

Definition2. (Singularity)

If a point z = α is called a singular point, or singularity of the complex function f(z) if f is not analytic at z = α , but every neighborhood $D_R(\alpha) = \{z: |z - \alpha| < R\}$ of α contains at least one point at which f(z) is analytic.

Example2.1

The function $f(z) = \frac{1}{1-z}$ is not analytic at z = 1, but is analytic for all other values of z. Thus the point z = 1 is a singular point of f (z).

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Example2.2

The function $f(z) = \frac{1}{z}$ is not analytic at z = 0, but is analytic for all other values of z. Thus the point z = 0 is a singular point of f (z).

Example2.3

Consider g (z) = log z , g (z) is analytic for all z except at the origin and at all points on the negative real-axis. Thus, the origin and each point on the negative real axis is a singularity of g (z).

Example2.4

The function $f(z) = \frac{1}{z(z-i)}$ is not analytic at z = 0 and z = *i*, but is analytic for all other values of z. Thus the point z = 0 and z = *i* are the singular points of f (z).

Definition3. (Isolated Singularity)

The point α is called an isolated singularity of the complex function f(z) if f is not analytic at z = α, but there exists a real number R > 0 such that f(z) is analytic everywhere in the punctured disk $D_R^*(\alpha) = \{z: 0 < |z - \alpha| < R\}$.

Example3.1

The function $f(z) = \frac{1}{1-z}$ has an isolated singularity at z = 1.

Example3.2

The function $f(z) = \frac{1}{z}$ has an isolated singularity at z = 0.

Example3.3

Consider the function $f(z) = \frac{(z+1)}{z^2(z^2+1)}$

$$= \frac{(z+1)}{z^2(z-i)(z+i)}$$ has three isolated singularities at z = 0, z = *i* and z = -*i*

Example3.4

The function g(z) = log z, however, the singularity at z = 0 (or at any point of the negative real axis) that is not isolated, because any neighborhood of contains points on the negative real axis, and g(z) = log z is not analytic at those points. Functions with isolated singularities have a Laurent series because the punctured disk $D_R^*(\alpha)$ is the same as the annulus A (α, 0, R). The logarithm function g (z) does not have a Laurent series at any point z = -a on the negative real-axis. We now look at this special case of Laurent's theorem in order to classify three types of isolated singularities.

2. REMOVABLE SINGULARITY, POLE OF ORDER K, ESSENTIAL SINGULARITY

Let f (z) has an isolated singularity at α with Laurent series expansion

$f(z) = \sum c_n (z - \alpha)^n$, valid for z Є A (α, 0, R) and $-\infty \le n \le \infty$.

Then we distinguish the following types of singularities at α.

Definition4. (Removable Singularities)

If $c_n = 0$ for n = -1, -2, -3…, then we say that f (z) has a removable singularity at α. That is no negative powers terms in the Laurent series expansion of f (z).

If f (z) has a removable singularity at z = α, then it has a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - \alpha)^n$, valid for z Є A (α, 0, R).

The power series for f (z) defines an analytic function in the disk $D_R(\alpha)$.

If we use this series to define $f(\alpha) = c_0$, then the function f (z) becomes analytic at z = α, removing the singularity.

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Example4.1

Consider the function $f(z) = \frac{sinz}{z}$. It is undefined at z = 0 and has an isolated singularity at z = 0, as the Laurent series for f (z) is

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$
$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots,$$

valid for |z| > 0.

We can remove this singularity if we define f (0) = 1, for then f (z) will be analytic

at z = 0

Example4.2

Consider $g(z) = \frac{cosz-1}{z^2}$, which has an isolated singularity at the point z = 0, as the Laurent series for g (z) is

$$g(z) = \frac{1}{z^2}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)$$
$$= -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \cdots,$$

valid for |z| > 0.

We can remove this singularity if we define f (0) = - 1/2, then g (z) will be analytic

for all z.

Definition5. (Pole of order k)

If k is a positive integer such that $c_{-k} \neq 0$ but $c_n = 0$ for n = -k-1, -k-2, -k-3…, then we say that f (z) has a pole of order k at α. That is in the Laurent series expansion of f (z) there are only (k terms) finite number of negative power terms. If f (z) has a pole of order k at z = α, the Laurent series for f (z) is

$f(z) = \sum_{n=-k}^{\infty} c_n (z - \alpha)^n$, valid for z Є A (α, 0, R) ,where $c_{-k} \neq 0$.

Example5.1

Consider the function $f(z) = \frac{sinz}{z^3}$

$$= \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$
$$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots,$$

Here the function f (z) has a pole of order k = 2 at z = 0.

Example5.2

Consider the function $f(z) = \frac{5z+1}{(z-2)^3(z+3)(z-2)}$

Has a pole of order 3 at z = 2 and simple poles at z = -3 and z = 2.

Definition5.1. (Simple Pole)

If f (z) has a pole of order 1 at z = α, we say that f (z) has a simple pole at z = α.

Example5.1.1

Consider the function $g(z) = \frac{e^z}{z}$

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$$= \frac{1}{z}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \right)$$

$$= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots,$$

Clearly the function g (z) has a simple pole at z = 0.

Definition6. (Essential Singularities)

If $c_n \neq 0$ for infinitely many negative integers n, then we say that f (z) has an essential singularity at $z = \alpha$. That is in the Laurent series expansion of f (z), there are infinite number of negative power terms.

Example6.1

Consider the function $f(z) = z^2 \sin(1/z)$

$$= z^2\left(\frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \frac{\left(\frac{1}{z}\right)^7}{7!} + \cdots\right)$$

$$= z - \frac{1}{3!}z^{-1} + \frac{1}{5!}z^{-3} - \frac{1}{7!}z^{-5} + \cdots,$$

Here the function f (z) has an essential singularity at the origin. Z = 0

Example6.2

Consider the function $f(z) = e^{\frac{1}{z}}$

$$= 1 + \frac{\frac{1}{z}}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots,$$

$$= 1 + \frac{1}{1!}z^{-1} + \frac{1}{2!}z^{-2} + \frac{1}{3!}z^{-3} + \cdots,$$

Here the function f (z) has an essential singularity at the origin. Z = 0

Definition7. (Zero of order k).

A function f (z) analytic in $D_R(\alpha)$ has a zero of order k at the point $z = \alpha$ if and only if $f^{(n)}(\alpha) = 0$ for n = 0,1,2,..., k-1, and $f^{(k)}(\alpha) \neq 0$ (k^{th} derivative of f(z))

Example7.1

In the following function

$f(z) = z \sin z^2$

$$= z^3 - \frac{1}{3!}z^7 + \frac{1}{5!}z^{11} - \frac{1}{7!}z^{15} + \cdots,$$

We have $f'(z) = 2z^2 \cos z^2 + \sin z^2$

$f''(z) = 6z \cos z^2 - 4z^3 \sin z^2$

$f'''(z) = 6 \cos z^2 - 8z^4 \cos z^2 - 24 z^2 \sin z^2$

Then, $f(0) = f'(0) = f''(0) = 0$, but $f'''(0) = 6 \neq 0$.

Hence the function f (z) has a zero of order k = 3 at z = 0.

Definition7.1. (Simple Zero).

If the function f (z) has a zero of order one, then we say that f (z) has a simple zero.

Example7.1.1

The function f (z) = z has a simple zero at z = 0

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We have $f'(z) = 1$, then $f'(0) = 1 \neq 0$, hence the function f (z) has zero of order one

Theorem I. A function f (z) analytic in $D_R(\alpha)$ has a zero of order k at the point $z = \alpha$ iff its Taylor series given by $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has $c_0 = c_1 = \cdots = c_{k-1} = 0$

and $c_k \neq 0$.

Proof.

Suppose f(z) is analytic and has a zero of order k at the point $z = \alpha$, then by Taylor's theorem

$f(z) = \sum_{n=0}^{\infty} c_n (z - \alpha)^n$ Where $c_n = \frac{f^{(n)}(\alpha)}{n!}$

Given f(z) has a zero of order k at $z = \alpha$, hence we have $c_0 = c_1 = \cdots = c_{k-1} = 0$ and $c_k \neq 0$

Conversely suppose the Taylor series of f (z) has $c_0 = c_1 = \cdots = c_{k-1} = 0$ and $c_k \neq 0$ which implies that from the definition of zero, f (z) has a zero of order k at $z = \alpha$

Theorem II. Suppose f (z) is analytic in $D_R(\alpha)$. Then f (z) has a zero of order k at the point $z = \alpha$ if and only if it can be expressed in the form $f(z) = (z - \alpha)^k g(z)$ where g (z) is analytic at $z = \alpha$ and $g(\alpha) \neq 0$.

Proof.

Suppose f (z) has a zero of order k at $z = \alpha$, then by Theorem 1, f (z) can be written as

$f(z) = c_k(z-\alpha)^k + c_{k+1}(z-\alpha)^{k+1} + \cdots$

$= (z-\alpha)^k\{ c_k + c_{k+1}(z-\alpha) + c_{k+2}(z-\alpha)^2 + \cdots \}$

$= (z-\alpha)^k g(z)$,

Where $g(z) = c_k + c_{k+1}(z-\alpha) + c_{k+2}(z-\alpha)^2 + \cdots$ which is analytic at $z = \alpha$ and $g(\alpha) \neq 0$

Corollary II.1

If f(z) and g(z) are analytic at $z = \alpha$, and have zeros of orders m and n, respectively at $z = \alpha$, then their product $h(z) = f(z)g(z)$ has a zero of order $m + n$ at $z = \alpha$.

Proof.

Suppose f(z) and g(z) are analytic at $z = \alpha$ and have zeros of orders m and n respectively at $z = \alpha$

Then by Theorem II, $f(z) = (z-\alpha)^m h_1(z)$, and $g(z) = (z-\alpha)^n h_2(z)$,

Where $h_1(z)$ and $h_2(z)$ are analytic at $z = \alpha$, $h_1(\alpha) \neq 0$, $h_2(\alpha) \neq 0$

Now $h(z) = f(z)g(z) = (z-\alpha)^m h_1(z) (z-\alpha)^n h_2(z) = (z-\alpha)^{m+n} h_1(z) h_2(z) = (z-\alpha)^{m+n} h_3(z)$

Where $h_3(z) = h_1(z) h_2(z)$ analytic at $z = \alpha$ and $h_3(\alpha) \neq 0$

Hence h (z) has a zero of order m+n at $z = \alpha$

Example II.1 Let $f(z) = z^3 \sin z$. Then f (z) can be factored as the product of z^3 and sinz, which have zeros of orders m = 3 and n = 1, respectively, at z = 0.

Hence z = 0 is a zero of order 4 of f (z).

Let $g(z) = z^3$ and $h(z) = \sin z$ and $f(z) = g(z) h(z)$

Clearly g (z) and h (z) have zeros of orders m = 3 and n =1 respectively at z = 0 and hence by Corollary II.1, f (z) has zero of order m +n = 3 + 1 = 4 at z = 0

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Theorem III. A function f (z) analytic in the punctured disk $D_R^*(\alpha)$ has a pole of order k at z = α if and only if it can be expressed in the form $f\ (z) = \frac{h(z)}{(z-\ \alpha)^k}$ where the function h (z) is analytic at the point z = α and h (α) ≠ 0.

Proof.

If f (z) has a pole of order k at z = α, the Laurent series for f (z) is

$f(z) = \sum_{n=-k}^{\infty} c_n (z - \alpha)^n$, valid for z Є A (α, 0, R), where $c_{-k} \neq 0$.

$f(z) = \sum_{n=0}^{\infty} c_n (z - \alpha)^n + \sum_{n=-1}^{-k} c_n (z - \alpha)^n$

$= \sum_{n=0}^{\infty} c_n (z - \alpha)^n + c_{-1}(z-\alpha)^{-1} + c_{-2}(z-\alpha)^{-2} + \ldots + c_{-k}(z-\alpha)^{-k}$

$= \sum_{n=0}^{\infty} c_n (z - \alpha)^n + (z-\alpha)^{-k} (c_{-1}(z-\alpha)^{k-1} + c_{-2}(z-\alpha)^{k-2} + \ldots + c_{-k})$

$= (z-\alpha)^{-k} \{(\sum_{n=0}^{\infty} c_n (z - \alpha)^n) (z-\alpha)^{-k} + c_{-1}(z-\alpha)^{k-1} + c_{-2}(z-\alpha)^{k-2} + \ldots + c_{-k})\}$

Let us take $h (z) = \{(\sum_{n=0}^{\infty} c_n (z - \alpha)^n) (z-\alpha)^{-k} + c_{-1}(z-\alpha)^{k-1} + c_{-2}(z-\alpha)^{k-2} + \ldots + c_{-k})\}$ which is analytic at z = α and h(α) ≠ 0

Therefore $f(z) = (z-\alpha)^{-k} h(z)$

That is $f\ (z) = \frac{h(z)}{(z-\ \alpha)^k}$

Similarly the converse is also true

The following Corollaries are useful in determining the order of a zero or a pole

Corollary III.1. If f (z) is analytic and has a zero of order k at the point z = α, then $g(z) = \frac{1}{f(z)}$ has a pole of order k at z = α.

Proof.

Suppose f (z) has a zero of order k at z = α, then by Theorem II, $f(z) = (z - \alpha)^k g(z)$

where g (z) is analytic at z = α and g(α) ≠ 0.

Now $\frac{1}{f(z)} = \frac{1}{(z-\alpha)^k g(z)}$

Let us take $h (z) = \frac{1}{g(z)}$, then $\frac{1}{f(z)} = \frac{h(z)}{(z-\alpha)^k}$, clearly h (z) is analytic at z=α and h(z) ≠ 0

By Theorem III, $\frac{1}{f(z)}$ has a pole of order k at z = α

Corollary III.2. If f (z) has a pole of order k at the point z = α, then $g(z) = \frac{1}{f(z)}$ has a zero of order k at z = α.

Proof.

If f (z) has a pole of order k at the point z = α, then by Theorem III, $f\ (z) = \frac{h(z)}{(z-\ \alpha)^k}$

Where h (z) is analytic at z = α and h (α) ≠ 0.

Now $\frac{1}{f(z)} = \frac{(z-\alpha)^k}{h(z)}$

Let us take $G (z) = \frac{1}{h(z)}$, then $\frac{1}{f(z)} = (z - \alpha)^k G(z)$, where G(z) is analytic at z = α and G(α) ≠ 0

By Theorem II, $\frac{1}{f(z)} = g(z)$ has a zero of order k at z = α, where $g (z) = (z - \alpha)^k G(z)$

Corollary III.3. If f(z) and g(z) have poles of orders m and n, respectively at the point z = α, then their product h(z) = f(z)g(z) has a pole of order m +n at z = α.

Proof.

Suppose f(z) and g(z) have poles of orders m and n, respectively at the point z = α, then by Theorem III, $f\ (z) = \frac{h_1(z)}{(z-\alpha)^m}$ and $g\ (z) = \frac{h_2(z)}{(z-\alpha)^n}$

Now h(z) = f(z)g(z) $= \frac{h_1(z)}{(z-\alpha)^m}\ \frac{h_2(z)}{(z-\alpha)^n} = \frac{H(z)}{(z-\alpha)^{m+n}}$, where H(z) = $h_1(z)h_2(z)$ which is analytic at z = α and H(α) ≠ 0.

Hence by Theorem III, h(z) has a pole of order m + n at z = α

Corollary III.4. Let f (z) and g (z) be analytic with zeros of orders m and n, respectively at z = α. Then their quotient $h(z) = \frac{f(z)}{g(z)}$ has the following behavior:

(i) If m > n, then h(z) has a zero of order m - n at z = α.

(ii) If m < n, then h(z) has a pole of order n - m at z = α.

(iii) If m = n, then h(z) has a removable singularity at z = α, and can be defined so that h(z) is analytic at z = α, by $h(\alpha) = \lim_{z \to \alpha} h(z)$.

Example III.1. Locate the zeros and poles of $h(z) = \frac{tanz}{z}$, and determine their order.

Given $h(z) = \frac{tanz}{z} = \frac{sinz}{zcosz} = \frac{f(z)}{g(z)}$

We know that the zeros of f (z) = sinz occur at the points z = nπ, where n is an integer. Because f ′ (nπ) = cosnπ ≠ 0, the zeros of f (z) are simple. Similarly, the function g (z) = zcosz has simple zeros at the points z = 0 and z = $\left(n+\frac{1}{2}\right)\pi$, where n is an integer. From the information given, we find that $h(z) = \frac{f(z)}{g(z)}$ behaves as follows:

i. h (z) has simple zeros at z = nπ, where n = ±1, ±2…

ii. h (z) has simple poles at z = $\left(n+\frac{1}{2}\right)\pi$, where n is an integer; and

iii. h (z) is analytic at z = 0 if we define $h(0) = \lim_{z \to 0} h(z) = 1$.

Example.III.2. Locate the poles of $g(z) = \frac{1}{5z^4+26z^2+5}$ and specify their order.

The roots of the quadratic equation $5z^2 + 26z + 5 = 0$ are z = - 5 and $z = -\frac{1}{5}$.

If we replace z with z^2 in this equation, the function $f(z) = 5z^4 + 26z^2 + 5$ has roots $z^2 = -5$ and $z^2 = -\frac{1}{5}$,

Therefore the roots of f (z) are $z = \pm i\sqrt{5}$ and $z = \pm\frac{i}{\sqrt{5}}$.

That is f (z) has simple zeros at the points $z = \pm i\sqrt{5}$ and $z = \pm\frac{i}{\sqrt{5}}$.

Corollary 3.1 implies that g(z) has simple poles at $z = \pm i\sqrt{5}$ and $z = \pm\frac{i}{\sqrt{5}}$.

Example III.3. Locate the zeros and poles of $g(z) = \frac{\pi cot\pi z}{z^2}$, and determine their order.

The function $f(z) = z^2 \sin\pi z$ has a zero of order k = 3 at z = 0 and simple zeros at the points z = ±1, ±2,…, Corollary implies that g(z) has a pole of order 3 at the point z = 0 and simple poles at the points z = ±1, ±2,…,

Example III.4. Find the poles of $f(z) = \frac{1}{sinz-cosz}$

Here the poles of f(z) are the zeros of sinz – cosz

Take sinz – cosz = 0 => sinz = cosz

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=> $\frac{sinz}{cosz} = 1$ => tanz = 1 => $z = n\pi + \frac{\pi}{4}$, n = 0,1,2….

Hence the simple zeros of sinz – cosz are $z = n\pi + \frac{\pi}{4}$, n = 0,1,2….

Therefore the simple poles of f (z) are $z = n\pi + \frac{\pi}{4}$, n = 0,1,2….

3. RESIDUE THEORY

Definition.1. (Residue). Let f (z) has a non-removable isolated singularity at the point z_0. Then f(z) has the Laurent series representation for all z in some punctured disk $D_R^*(z_0)$ given by $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$

The coefficient a_{-1} of $\frac{1}{z-z_0}$ is called the residue of f(z) at z_0.

It is denoted by $Res[f,z_0] = a_{-1}$

Example .1.1.

Consider f (z) = $e^{2/z}$

Then the Laurent series of f about the point z_0 = 0is given by

$$= 1 + \frac{2}{1!\,z} + \frac{2^2}{2!\,z^2} + \frac{2^3}{3!\,z^3} + \cdots,$$

The co-efficient of $\frac{1}{z-z_0} = \frac{1}{z-0} = \frac{1}{z}$ is 2

Hence by definition of residue, residue of f (z) = $e^{2/z}$ at z_0 = 0 is given by Res [f, z_0] = 2

Example .1.2. Find residue of f (z) = $\frac{3}{2z+z^2-z^3}$ at $z_0 = 0$

f(z) = $\frac{3}{z(2+z-z^2)} = \frac{3}{z(z+1)(2-z)}$

Now $\frac{3}{z(z+1)(2-z)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{2-z}$

⇒A(z+1)(2-z) + Bz(2-z) + Cz (z+1) = 3

⇒A($-z^2$ +2 + z) +B(2z –z^2) +C(z^2 +z) = 3

⇒-A –B +C = 0---- (a)

A +2B +C = 0 ---- (b)

2A =3 ---- (c) ⇒A = 3/2

(a) ⇒-B + C = A= 3/2

(b) ⇒2B + C = -A = -3/2

Adding B + 2C =0⇒ B = - 2C

Put B = - 2C in -B + C = 3/2

⇒3C = 3/2⇒C= 1/2

Put C= 1/2, B = -2C = -1⇒ B = -1

Hence f(z) = = $\frac{A}{z} + \frac{B}{z+1} + \frac{C}{2-z} = \frac{3}{2z} - \frac{1}{z+1} + \frac{1}{2(2-z)}$

$= \frac{3}{2z} - (1+z)^{-1} + \frac{1}{2}(2-z)^{-1} = \frac{3}{2z} - (1- z+ z^2 -\ldots) + \frac{1}{2} 2^{-1}\left(1 - \frac{z}{2}\right)^{-1}$

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$= \frac{3}{2z}$ - (1- z+ z^2 -….) $+\frac{1}{4}$ $(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + ….)$

$= \frac{3}{2z} - \frac{3}{4} + \frac{9z}{8}$- …..

The residue of f at 0 is given by Res [f,0] = coefficient of $\frac{1}{z} = \frac{3}{2}$

Example 1.3. Find residue of f (z) = $\frac{e^z}{z^3}$ at $z_0 = 0$

Laurent expansion of f(z) = $\frac{1}{z^3}\{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} ...\}$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z2!} + \frac{1}{3!} ...$$

The residue of f at 0 is given by Res [f,0] = coefficient of $\frac{1}{z} = \frac{1}{2}$

Contour integration

Contour integration is the process of calculating the values of a contour integral around a given contour in the complex plane.

The Cauchy integral formulae are useful in evaluating contour integrals over a simple closed contour C where the integrand has the form $\frac{f(z)}{(z-z_0)^k}$ and f is an analytic function

Example 1.

Evaluate $\int_C ydz$ along the curve C : x = t-1, y = e^{t-1}, 2< t < 3

Solution.

Let z = x+iy ⇒ dz = dx + i dy

Given curve x = t-1 ⇒ dx = dt

And y = e^{t-1} ⇒ dy = e^{t-1}dt

$$\int_C ydz = \int_2^3 e^{t-1}(dx + idy) = \int_2^3 e^{t-1}(dt + ie^{t-1}dt) = \int_2^3 e^{t-1}dt + \int_2^3 ie^{2(t-1)}dt$$

$$= e^{t-1}\Big]_2^3 + \frac{ie^{2t-2}}{2}\Big]_2^3 = e^2 - e + \frac{ie^4 - e^2}{2} =$$

Example 2.

If C is the curve y = x^3-3x^2+4x-1 joining the points (1,1) and (2,3) then

find the value of $\int_C(12z^2 - 4iz)dz$

$$\int_C(12z^2 - 4iz)dz = \int_{1+i}^{2+3i}(12z^2 - 4iz)dz = \frac{12z^3}{3} - \frac{4iz^2}{2}\Big]_{1+i}^{2+3i}$$

= 4(2+3i)^3 – 2i(2+3i)^2 - 4(1+i)^3 + 2i(1+i)^2 = -156 +38i

Recall (i).(Cauchy's integral Theorem)

Let D be any simply connected domain. Let C be any closed contour contained in D and f(z) analytic in D, then $\oint_C f(z)\, dz = 0$

Recall (ii).

For a function f(z) analytic in $D_R^*(z_0)$and for any r with 0 < r < R, the Laurent series

coefficients of f(z) are given by

$a_n = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi$for n =0,±1 ,±2, ….. ------(I)

WhereC denotes the circle {z: $|z-z_0| = r$} with positive orientation.

Put n = -1 in Equation (I) and replace C with any positively oriented simple closed contour C

containingz_0, provided z_0 is the still only singularity of f(z) that lies inside C,

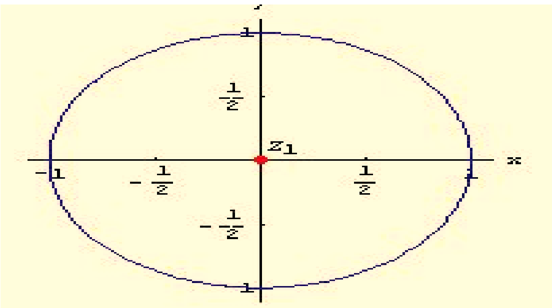
Then we obtain $a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\, dz.$

We know that a_{-1} is the Res[f,z_0]⇒ Res[f,z_0] = $\frac{1}{2\pi i}\oint_C f(z)\, dz$

$\Rightarrow \oint_C f(z)\, dz = 2\pi i$ Res[f,z_0]

If we know the Laurent series expansion for f(z), then using above equation we can evaluate contour integrals.

Example: ii.1. Evaluate $\oint_C e^{\frac{2}{z}}\, dz$where C denotes the circle C = {z: |z| =1}with positive orientation.



Solution.

Let f(z) = $e^{\frac{2}{z}}$

From Example.1, we have Res [f, 0] = 2

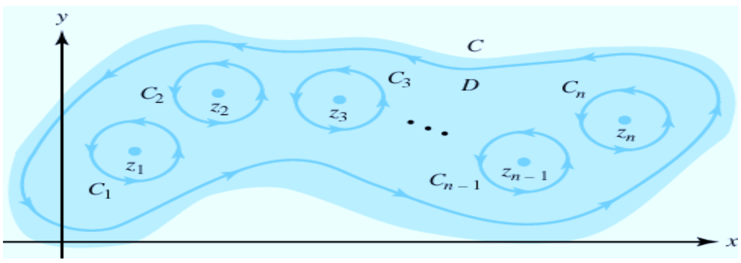
Recall 2. gives us $\oint_C f(z)\, dz = 2\pi i$ Res[f,z_0]

Hence $\oint_C e^{\frac{2}{z}}\, dz = 2\pi i$ Res[f,0] $= 2\pi i\ (2) = 4\pi i$

Theorem 1 (Cauchy's Residue Theorem).

Let D be a simply connected domain, and let C ⊂D be a closed positively oriented contour within and on the functionf(z) is analytic, except finite number of singular $z_1, z_2, \ldots, z_n$, then

$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n}$ Res[f, z_k]



Proof.

Let C_i be the neighborhood of z_i, (i=1,2…n) lies inside C such that all $C_i$are disjoint.

Since each z_i is a singular point of f and each C_i is a neighborhood of corresponding z_i(i=1,2,..n), f is analytic in and on C except these neighborhoods C_i (i=1,2…n).

Then by Cauchy's Theorem,(Recall 1)

$\oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \ldots\ldots - \oint_{C_n} f(z)\,dz = 0$

$\Rightarrow \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \ldots\ldots + \oint_{C_n} f(z)\,dz$

$\Rightarrow \oint_C f(z)\,dz = 2\pi i Res[f,z_1] + \ldots.. + 2\pi i\, Res[f,z_n]$ (by Recall 2)

$\Rightarrow \oint_C f(z)\,dz = 2\pi i \sum_{k=0}^{n} Res[f, z_k]$

Note 1.1: The residue at $z_0$depend only the coefficient a_{-1}in the Laurent expansion, if f(z) has a

removable singularity at z_0, then the Laurent expansion has no negative power term and

hence$a_{-1}=0 \Rightarrow Res[f,z_0] = 0$.

Theorem 2.(Residues at Poles).

(i) If f(z) has a simple pole at z_0, then $Res[f,z_0] = \lim_{z \to z_0} (z - z_0) f(z)$

(ii) If f(z) has a pole of order 2 at z_0, then $Res[f,z_0] = \lim_{z \to z_0} \frac{d}{dz}((z - z_0)^2 f(z))$

(iii) If f(z) has a pole of order 3 at z_0, then $Res[f,z_0] = \frac{1}{2!} \lim_{z \to z_0} \frac{d^2}{dz^2}((z - z_0)^3 f(z))$

(v) If f(z) has a pole of order k at z_0,

then $Res[f,z_0] = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}}((z - z_0)^k f(z))$

Proof.

i) Suppose f(z) has a simple pole at $z = z_0$, then the Laurent series expansion

$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + a_{-1}(z - z_0)^{-1}$

$\Rightarrow (z - z_0)f(z) = (z-z_0) \sum_{n=0}^{\infty} a_n (z - z_0)^n + a_{-1}$

$\Rightarrow (z - z_0)f(z) = (z-z_0) \sum_{n=0}^{\infty} a_n (z - z_0)^n + Res[f,z_0]$

Taking $\lim z \to z_0$, both sides

$\lim_{z \to z0} (z - z_0)f(z) = \lim_{z \to z0} (z-z_0) \sum_{n=0}^{\infty} a_n (z - z_0)^n + \lim_{z \to z0} Res[f,z_0]$

$= 0 + Res[f,z_0]$

Hence $Res[f,z_0] = \lim_{z \to z0} (z - z_0)f(z)$

v) Suppose f(z) has a pole of order k at $z = z_0$, then the Laurent series expansion

$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + a_{-1}(z - z_0)^{-1} + a_{-2}(z - z_0)^{-2} + \ldots\ldots + a_{-k}(z - z_0)^{-k}$

Multiply both sides by $(z-z_0)^k$

$(z-z_0)^k f(z) = a_{-k} + \ldots.. + a_{-1}(z - z_0)^{k-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^n (z - z_o)^k$

Differentiate both sides k-1 times with respect to z,

$$\frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z) = 0+0+\ldots..+ a_{-1}(k-1)! + \frac{d^{k-1}}{dz^{k-1}}\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n (z-z_o)^k\right)$$

$$= a_{-1}(k-1)! + a_0(z-z_0)k! + a_1(z-z_0)^2\left(\frac{(k+1)!}{2!}\right) + \ \ldots.$$

Taking lim $z \to z_0$, both sides

$$\lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z) = \lim_{z \to z_0} a_{-1}(k-1)! + 0 + \ldots$$

$$= a_{-1}(k-1)! = \text{Res}\,[f,z_0]\,(k-1)!$$

Hence Res $[f,z_0] = \frac{1}{(k-1)!}\lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z)$

ii) and iii) are the particular case of v) (take k = 2 and k= 3)

Example 2.1.

Find residue of f (z) = $\frac{e^z}{z^2-1}$ at $z_0 = 1$

Solution.

Given f (z) = $\frac{e^z}{z^2-1} = \frac{e^z}{(z-1)(z+1)}$

The poles of f(z) are z = 1 , z =-1 (simple poles)

Res [f,1] = $\lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \frac{(z-1)\, e^z}{(z-1)(z+1)} = \lim_{z \to 1} \frac{e^z}{(z+1)} = \frac{e}{2}$

Example 2.2 .

Find the residue of f(z) = $(z^8 - \omega^8)^{-1}$, where ω is any complex constant.

Solution.

Given f(z) = $(z^8 - \omega^8)^{-1} = \frac{1}{(z^8-\omega^8)}$

The poles of f(z) are the zeros of $z^8-\omega^8 \Rightarrow$ zeros are given by $z^8-\omega^8 = 0$

$\Rightarrow z^8 = \omega^8 \Rightarrow z^8 = \omega^8(\cos 2n\pi + i\sin 2n\pi)$, n = 0,1,2,…7 $\Rightarrow z^8 = \omega^8 e^{2ni\pi}$

$\Rightarrow z = \omega\, e^{2n\pi i/8} \Rightarrow z = \omega\, e^{n\pi i/4}$, n = 0,1,2,…7

Hence $z = \omega\, e^{n\pi i/4}$, n = 0,1,2,…7 are the simple poles of f(z)

Let $a_n = \omega\, e^{n\pi i/4}$, n = 0,1,2,…7

The residue of f (z) at $z = a_n$, is given by Res $[f, a_n] = \lim_{z \to a_n} (z-a_n) f(z)$

$$= \lim_{z \to a_n} (z-a_n) \frac{1}{(z^8-{a_n}^8)}$$

Since it is not easy to factories $(z^8 - a_n^8)$ into eight factors , so we have to use L'Hospital's rule

(that is differentiating Nr and Dr separately w.r.to z)

$$= \lim_{z \to a_n} \frac{1}{8z^7} = \frac{1}{8a_n^8}, \ n = 0,1,\ldots.7.$$

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Example 2.3

Find the residue of $\frac{1}{sinh\pi z}$

Solution.

Given f(z) = $\frac{1}{sinh\pi z}$

The poles of f(z) are the zeros of sinhπz,

Also the zeros of sinhπz are z = ni, for all integer n (since sinhnπi = 0 for all n)

Hence Res [f,ni] = $\lim_{z \to ni} (z - ni) \frac{1}{sinh\pi z}$

By using L'Hospital's rule

Res [f,ni] = $\lim_{z \to ni} \frac{1}{\pi cosh\pi z} = \frac{1}{\pi cosh\pi ni} = \frac{1}{\pi cos\pi n}$ (Since coshix= cosx)

= $(-1)^n/\pi$ (Since

Example 2.3. Find the residue of f(z) = $\frac{\pi \cot(\pi z)}{z^2}$ at $z_0 = 0$

Solution.

Given f(z) = $\frac{\pi \cot(\pi z)}{z^2} = \frac{\pi \cos(\pi z)}{z^2 \sin(\pi z)}$

Since z^2 has a zero of order 2 at $z_0 = 0$ and sin(πz) has a simple zero 1 at $z_0 = 0$, we have $z^2\sin(\pi z)$ has a zero of order 3at $z_0 = 0$ and $\pi\cos(\pi z) \neq 0$.

Hence f(z) has a pole of order 3 at $z_0 = 0$.

By part (iii) of Theorem .2, we have

Res [f,0] = $\frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}((z-0)^3 f(z))$

$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}(z^3 \frac{\pi \cot(\pi z)}{z^2})$

$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}(z \; \pi\cot(\pi z)) = \frac{1}{2!} \lim_{z \to 0} \frac{d}{dz}(\pi \cot(\pi z) - \pi z csc^2(\pi z) \pi)$

$= \frac{1}{2!} \lim_{z \to 0} \frac{d}{dz}(\pi \cot(\pi z) - \pi^2 z csc^2(\pi z))$

$= \frac{1}{2!} \lim_{z \to 0} (- \pi \csc^2(\pi z) \pi - \pi^2 csc^2(\pi z) + \pi^2 z 2 \csc(\pi z)\csc(\pi z)\cot(\pi z)\pi)$

$= \frac{1}{2!} \lim_{z \to 0} (- 2\pi^2 \csc^2(\pi z) + 2\pi^3 z \csc^2(\pi z)\cot(\pi z))$

$= \frac{2\pi^2}{2!} \lim_{z \to 0} \csc^2(\pi z) (- 1 + \pi z \cot(\pi z))$

$= \pi^2 \lim_{z \to 0} \frac{1}{\sin^2(\pi z)}(- 1 + \pi z \frac{\cos(\pi z)}{\sin(\pi z)})$

Res [f,0] $= \pi^2 \lim_{z \to 0} \frac{(\pi z cos(\pi z) - \sin(\pi z))}{\sin^3(\pi z)}$

As lim z → 0, LHS is indeterminate, so we have to use L'Hospital's rule to evaluate the limit

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(that is differentiating Nr and Dr separately w.r.to z)

$$\text{Res}\,[f,0] = \pi^2 \lim_{z \to 0} \frac{(\pi cos(\pi z) - \pi z sin(\pi z)\pi - \pi\cos(\pi z))}{3\sin^2(\pi z)\cos(\pi z)\pi}$$

$$= \pi^2 \lim_{z \to 0} \frac{-\pi^2 z sin(\pi z)}{3\pi\sin^2(\pi z)\cos(\pi z)}$$

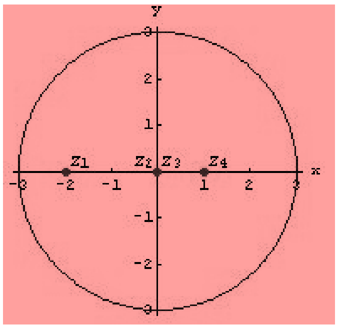
$$= \pi^2 \lim_{z \to 0} \frac{-\pi z}{3\sin(\pi z)\cos(\pi z)}$$

$$= \frac{\pi^2}{3} \lim_{z \to 0} \frac{-\pi z}{\sin(\pi z)\cos(\pi z)} = \frac{-\pi^2}{3} \lim_{z \to 0} \left(\frac{\pi z}{\sin(\pi z)}\right) \lim_{z \to 0} \left(\frac{1}{\cos(\pi z)}\right)$$

$$= \frac{-\pi^2}{3} \frac{1}{\lim_{z\to 0}\left(\frac{\sin(\pi z)}{\pi z}\right)} \lim_{z \to 0} \left(\frac{1}{\cos(\pi z)}\right) = \frac{-\pi^2}{3}(1)(1) = \frac{-\pi^2}{3}$$

Example 2.4.

Find $\int_C \frac{dz}{z^4+z^3-2z^2}$ where C denotes the circle {z: |z| =3} with positive orientation.



Solution.

Let $f(z) = \frac{1}{z^4+z^3-2z^2} = \frac{1}{z^2(z^2+z-2)} = \frac{1}{z^2(z+2)(z-1)}$

The singularities of f(z) that lie inside C are simple poles at the points z =1 and z= -2, and a pole of order 2 at z =0.

To find the Residue at z = 0 :

$$\text{Res}\,[f,0] = \lim_{z \to 0} \frac{d}{dz}((z-0)^2 f(z)) = \lim_{z \to 0} \frac{d}{dz}\left(\frac{z^2}{z^2(z+2)(z-1)}\right)$$

$$= \lim_{z \to 0} \frac{d}{dz}\left(\frac{1}{(z+2)(z-1)}\right) = \lim_{z \to 0} \frac{d}{dz}\left(\frac{1}{z^2+z-2}\right)$$

$$= \lim_{z \to 0} \frac{-2z-1}{(z^2+z-2)^2} = -\frac{1}{4}$$

To find the Residue at z = 1 :

$$\text{Res}\,[f,1] = \lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} (z-1) \frac{1}{z^2(z+2)(z-1)}$$

$$= \lim_{z \to 1} \frac{1}{z^2(z+2)} = \frac{1}{3}$$

To find the Residue at z = -2 :

$$\text{Res}\,[f,-2] = \lim_{z \to -2} (z+2) f(z) = \lim_{z \to -2} (z+2) \frac{1}{z^2(z+2)(z-1)}$$

$$= \lim_{z \to -2} \frac{1}{z^2(z-1)} = \frac{-1}{12}$$

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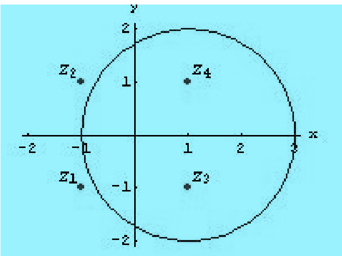
By Cauchy's residue theorem $\oint_C f(z)\, dz \;\; = 2\pi i \sum_{k=0}^{n} Res[f, z_k]$

$\int_C \frac{dz}{z^4+z^3-2z^2}$ = 2πi(Res[f,0] +Res[f,1]+Res[f,-2]) = $2\pi i(\frac{-1}{4} + \frac{1}{3} - \frac{1}{12})$ = 0

Example 2.5. Find $\int_C \frac{dz}{z^4+4}$ where C denotes the circle {z: |z-1| =2 } with positive orientation



Solution.

Let f(z) = $\frac{1}{z^4+4}$

To find the poles of f(z) , we know that poles of f(z) is nothing but the zeros of z^4 +4

Now we have to find the zeros of z^4 +4

Put $z^4 + 4 = 0 \Rightarrow z^4 = -4 = 4i^2 = (2i)^2 \Rightarrow z^2 = \pm 2i$

Let z = a+ib ⇒ $z^2 = (a+ib)^2 = a^2 + 2iab - b^2$

Suppose$z^2 = 2i \Rightarrow a^2 - b^2 + 2iab = 2i \Rightarrow a^2 - b^2 = 0$ and ab = 1

⇒ $a^2 = b^2$and b = 1/a

⇒ a = ±b and b = 1/a

If a = b , then b = 1 ⇒ a,b=1

If a = -b , then b = -1⇒a = 1 , b = -1

The zeros are z = a+ib , 1+i, 1-i

Suppose $z^2 = -2i \Rightarrow a^2 - b^2 + 2iab = -2i \Rightarrow a^2 - b^2 = 0$ and ab = -1

⇒ $a^2 = b^2$ and b = -1/a

⇒ a = ±b and b = -1/a

If a = b , then b = -1 ⇒ a,b=-1

If a = -b , then b = 1⇒a = -1 , b =1

The zeros are z = a+ib ,-1-i, -1+i

Hence the poles of f(z) are 1±i, -1±i (simple poles)

The poles lie inside the circle {z: |z-1| =2} with positive orientation are 1±i

$$Res[f,1+i] = \lim_{z \to 1+i} (z-(1+i))\, f(z) = \lim_{z \to 1+i} (z-(1+i))\left(\frac{1}{z^4+4}\right)$$

As lim z →1+i, LHS is indeterminate, so we have to use L'Hospital's rule to evaluate the limit

(that is differentiating Nr and Dr separately w.r.to z)

$$= \lim_{z \to 1+i} \left(\frac{1}{4z^3}\right) = \lim_{z \to 1+i} \left(\frac{z}{4z^4}\right) = \frac{1+i}{4(1+i)^4} = \frac{1+i}{4(-4)} = \frac{1+i}{-16}$$

Similarly

Res[f,1-i] = $\frac{1-i}{-16}$

By Cauchy's residue theorem $\oint_C f(z)\, dz \quad = 2\pi i \sum_{k=0}^{n} Res[f, z_k]$

$\int_C \frac{dz}{z^4+4}$ = 2πi(Res[f,1+i] +Res[f,1-i]) = 2πi($\frac{1+i}{-16} + \frac{1-i}{-16}$) = - $\frac{\pi i}{4}$

Result 3.

Let P(z) be a polynomial of degree at most 2. If a ,b and c are distinct complex numbers, then f(z) = $\frac{P(z)}{(z-a)(z-b)(z-c)}$ = $\frac{A}{(z-a)} + \frac{B}{(z-b)} + \frac{C}{(z-c)}$

Where A = Res [f,a] = $\frac{P(a)}{(a-b)(a-c)}$

B = Res [f,b] = $\frac{P(b)}{(b-a)(b-c)}$

C = Res [f,c] = $\frac{P(c)}{(c-a)(c-b)}$

Example 3.1.

Find the residue of f(z) = $\frac{3z+2}{z(z-1)(z-2)}$ and express f(z) in partial fractions.

Solution.

In Result I, take a= 0, b = 1, c= 2 and P(z) = 3z+2.

The residues are

A = Res[f,0] = $\frac{P(0)}{(0-1)(0-2)}$ = 1

B = Res [f,1] =$\frac{P(1)}{(1-0)(1-2)}$ = -5

C = Res [f,2] = $\frac{P(2)}{(2-0)(2-1)}$ = 4

The partial fraction expression of f(z)is given by

f(z) = $\frac{A}{(z-a)} + \frac{B}{(z-b)} + \frac{C}{(z-c)}$

= $\frac{1}{(z-0)} + \frac{-5}{(z-1)} + \frac{4}{(z-2)}$

= $\frac{1}{z}$ - $\frac{5}{(z-1)} + \frac{4}{(z-2)}$

Example 3.2. Find the residue of f (z) = $\frac{1}{z^4-1}$ and express in partial fractions.

f (z) = $\frac{1}{z^4-1}$ = $\frac{1}{(z^2-1)(z^2+1)}$ = $\frac{1}{(z-1)(z+1)(z+i)(z-i)}$ = $\frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{z-i} + \frac{D}{z+i}$

Where A = Res[f,1] = $\lim_{z \to 1} (z-1)\, f(z)$ = $\lim_{z \to 1} (z-1)\, \frac{1}{(z-1)(z+1)(z+i)(z-i)}$

= $\frac{1}{2(1+i)(1-i)}$ = $\frac{1}{2(1-(-1))}$ = $\frac{1}{4}$

B= Res[f,-1] = $\lim_{z \to -1} (z+1)\, f(z)$ = $\lim_{z \to -1} (z+1)\, \frac{1}{(z-1)(z+1)(z+i)(z-i)}$

= $\frac{1}{-2(-1+i)(-1-i)}$ = $\frac{1}{2(-1-1))}$ = $-\frac{1}{4}$

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C= Res [f,i] = $\lim_{z \to i}(z-i)\ f(z)$ $= \lim_{z \to i}(z-i)\ \frac{1}{(z-1)(z+1)(z+i)(z-i)}$

$= \frac{1}{2i(i+1)(i-1)} = \frac{1}{2i(-1-1)} = -\frac{1}{4i}$

D=Res[f,-i] = = $\lim_{z \to -i}(z+i)\ f(z)$ $= \lim_{z \to -i}(z+i)\ \frac{1}{(z-1)(z+1)(z+i)(z-i)}$

$= \frac{1}{-2i(-i+1)(-i-1)} = \frac{1}{2i(1-(-1))} = \frac{1}{4i}$

$$f(z) = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{z-i} + \frac{D}{z+i} = \frac{1}{4(z-1)} - \frac{1}{4(z+1)} - \frac{1}{4i(z-i)} + \frac{1}{4i(z+i)}$$

Result 4.

If a repeated root occurs in partial fraction, and P(z) has degree of at most 2, then f(z) = $\frac{P(z)}{(z-a)^2(z-b)} = \frac{A}{(z-a)^2} + \frac{B}{(z-a)} + \frac{C}{(z-b)}$

Where A = Res [(z-a)f(z),a]

B = Res [f, a]

C = Res [f, b]

Example 4.1. Find the residue of f(z) = $\frac{z^2+3z+2}{z^2(z-1)}$ and express in partial fraction.

Solution.

In Result II, take a = 0, b = 1 and P(z) = $z^2 + 3z + 2$, we have

f(z) = $\frac{P(z)}{(z-0)^2(z-1)} = \frac{A}{(z-0)^2} + \frac{B}{(z-0)} + \frac{C}{(z-1)}$

Where A = Res [(z-0)f(z),0] = Res [z $\frac{z^2+3z+2}{z^2(z-1)}$,0] = Res [$\frac{z^2+3z+2}{z\,(z-1)}$,0]

$= \lim_{z \to 0}\ (z-0)(\frac{z^2+3z+2}{z(z-1)}) = \lim_{z \to 0}\ (\frac{z^2+3z+2}{(z-1)})$ = -2

B = Res [f, 0] = $\lim_{z \to 0} \frac{d}{dz}\ (z-0)^2 \frac{z^2+3z+2}{z^2(z-1)} = \lim_{z \to 0} \frac{d}{dz}\ z^2 \frac{z^2+3z+2}{z^2(z-1)} = \lim_{z \to 0} \frac{d}{dz} \frac{z^2+3z+2}{(z-1)}$

$= \lim_{z \to 0} \frac{[(2z+3)(z-1)-(1)(z^2+3z+2)]}{(z-1)^2} = \lim_{z \to 0} \frac{[(z^2-2z-5)]}{(z-1)^2}$ = -5

C = Res [f, 1] = Res [$\frac{z^2+3z+2}{z^2(z-1)}$,1] = $\lim_{z \to 1}\ (z-1)(\frac{z^2+3z+2}{z^2(z-1)})$

$= \lim_{z \to 1}\ (\frac{z^2+3z+2}{z^2})$ = 6

f(z) = $\frac{-2}{(z-0)^2} + \frac{-5}{(z-0)} + \frac{6}{(z-1)} = \frac{-2}{z^2} + \frac{-5}{z} + \frac{6}{(z-1)}$

Example 4.2. Find the residue of f(z) = $\frac{1}{(z-1)^2(z-3)}$

Take P(z) = 1, a= 1, b=3

f(z) = $\frac{P(z)}{(z-1)^2(z-3)} = \frac{A}{(z-1)^2} + \frac{B}{(z-1)} + \frac{C}{(z-3)}$

A = Res [(z-1)f(z),1] = Res [(z-1) $\frac{1}{(z-1)^2(z-3)}$, 1] = $\lim_{z \to 1}\ (z-1)(\frac{1}{(z-1)(z-3)})$

$= \frac{1}{(1-3)} = \frac{1}{-2}$

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B = Res [f,1] = $\lim_{z \to 1} \frac{d}{dz}(z-1)^2 \frac{1}{(z-1)^2(z-3)} = \lim_{z \to 1} \frac{-1(1)}{(z-3)^2} = \frac{-1}{4}$

C = Res [f,3] = $\lim_{z \to 3}(z-3)\left(\frac{1}{(z-1)^2(z-3)}\right) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{4}$

4. EVALUATION OF REAL DEFINITE INTEGRALS

Cases of poles are not on the real axis.

Type I

Evaluation of the integral $\int_0^{2\pi} f(cos\theta sin\theta)d\theta$ where f(cosθ sinθ) is a real rational function of sinθ,cosθ.

First we use the transformation z = $e^{i\theta}$ = cosθ + i sinθ ------ (a)

And $\frac{1}{z} = \frac{1}{e^{i\theta}}$ = $e^{-i\theta}$ = = cosθ - i sinθ -------- (b)

From (a) and (b) , we have cosθ = $\frac{1}{2}(z+\frac{1}{z})$, sinθ = $\frac{1}{2i}(z-\frac{1}{z})$

Now z = $e^{i\theta}$ ⇒ dz = $ie^{i\theta}d\theta$ ⇒ dθ = $\frac{dz}{iz}$

Hence $\int_0^{2\pi} f(cos\theta sin\theta)d\theta = \int_C f[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}(z-\frac{1}{z})]\frac{dz}{iz}$

Where C, is the positively oriented unit circle |z| = 1

The LHS integral can be evaluated by the residue theorem and

$\int_C f[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}(z-\frac{1}{z})]\frac{dz}{iz}$ = 2πi ∑ Res(z_i) , where z_i is any pole in the interior of the circle |z| =1

Example I.1.

Evaluate $\int_0^{2\pi} e^{-cos\theta}\cos(n\theta+sin\theta)\,d\theta, where\ n\ is\ a\ positive\ integer.$

Solution.

Let I = $\int_0^{2\pi} e^{-cos\theta}[\cos(n\theta+sin\theta) - isin(n\theta+sin\theta)]\,d\theta$

$= \int_0^{2\pi} e^{-cos\theta} e^{-i(n\theta+sin\theta)}d\theta = \int_0^{2\pi} e^{-cos\theta-isin\theta}\, e^{-i(n\theta)}d\theta$

$= \int_0^{2\pi} e^{-(cos\theta+isin\theta)}\, e^{-i(n\theta)}d\theta = \int_0^{2\pi} e^{-e^{i\theta}}\, e^{-i(n\theta)}d\theta$

Let z = $e^{i\theta}$, dz = $ie^{i\theta}$ dθ ⇒ dθ = $\frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$ and C denotes the unit circle |z| = 1

Therefore I = $\int_C \frac{e^{-z}}{z^n}\left(\frac{dz}{zi}\right) = \frac{1}{i}\int_C \frac{e^{-z}\,dz}{z^{n+1}} = \int_C f(z)dz$ where f(z) = $\frac{e^{-z}}{iz^{n+1}}$

By Cauchy's residue theorem, $\int_C f(z)dz$ = 2πi∑ Res[f,z_k] where z_k are the singularities(poles) of f(z)

To find the poles of f(z) :

Since poles of f(z) = to the zeros of iz^{n+1} ,and the only zero of iz^{n+1} is z = 0 of order n+1

Hence the pole of f(z) is z =0 of order n+1

There are no poles on the real axis

To find the residue of f(z):

Res [f,0] = $\frac{1}{n!}\lim_{z \to 0}\frac{d^n}{dz^n}(z-0)^{n+1}f(z) = \frac{1}{n!}\lim_{z \to 0}\frac{d^n}{dz^n}(z-0)^{n+1}\frac{e^{-z}}{iz^{n+1}}$

$= \frac{1}{n!}\lim_{z \to 0}\frac{d^n}{dz^n}z^{n+1}\frac{e^{-z}}{iz^{n+1}} = \frac{1}{n!}\lim_{z \to 0}\frac{d^n}{dz^n}\frac{e^{-z}}{i} = \frac{1}{n!}\lim_{z \to 0}(-1)^n\frac{e^{-z}}{i}$

$= \frac{(-1)^n}{n!i}$

Hence ∑ Res[f,z_k] = Res[f,0]= $\frac{(-1)^n}{n!i}$

Therefore $\int_C f(z)dz = 2\pi i\sum \text{Res}[f,z_k] = 2\pi i \frac{(-1)^n}{n!i} = 2\pi \frac{(-1)^n}{n!}$

$\Rightarrow I = 2\pi \frac{(-1)^n}{n!}$

$\Rightarrow \int_0^{2\pi} e^{-cos\theta} [\cos(n\theta + sin\theta) - isin(n\theta + sin\theta)]\, d\theta = 2\pi \frac{(-1)^n}{n!}$

$\Rightarrow \int_0^{2\pi} e^{-cos\theta} \cos(n\theta + sin\theta)\, d\theta - i\int_0^{2\pi} e^{-cos\theta} \; sin(n\theta + sin\theta)]\, d\theta = 2\pi \frac{(-1)^n}{n!}$

Equating real and imaginary parts,

$\int_0^{2\pi} e^{-cos\theta} \cos(n\theta + sin\theta)\, d\theta = 2\pi \frac{(-1)^n}{n!}$

And $\int_0^{2\pi} e^{-cos\theta} \; sin(n\theta + sin\theta)]\, d\theta = 0$

Example I.2.

Prove that $\int_0^{2\pi} \frac{d\theta}{a+bcos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, a >b >0.

Solution.

Let I = $\int_0^{2\pi} \frac{d\theta}{a+bcos\theta}$

Put z = $e^{i\theta}$ $\Rightarrow d\theta = \frac{dz}{iz}$ and let C denotes the unit circle |z| = 1

Since z = $e^{i\theta}$ = cosθ + isinθ and $\frac{1}{z}$ = cosθ – isinθ, we have cosθ = $\frac{1}{2}$ (z + $\frac{1}{z}$)

$I = \int_0^{2\pi} \frac{d\theta}{a+bcos\theta} = \frac{1}{i}\int_C \frac{dz}{z\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)} = \frac{1}{i}\int_C \frac{dz}{za+\frac{z^2b}{2}+\frac{b}{2}} = \frac{2}{bi}\int_C \frac{dz}{z\frac{2a}{b}+z^2+1}$

$= \int_C f(z)dz$ where $f(z) = \frac{2}{bi(z^2+\frac{2az}{b}+1)}$

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i\sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

To find the poles of f(z):

Poles of f(z) = to the zeros of $bi(z^2+\frac{2az}{b}+1)$ and the zeros are given by $bi(z^2+\frac{2az}{b}+1) = 0$

$\Rightarrow (z^2+\frac{2az}{b}+1) = 0$ -------- (a)

$\Rightarrow z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}}{2} = \frac{-\frac{2a}{b} \pm 2\sqrt{a^2-b^2}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$

$\Rightarrow z = \frac{-a+\sqrt{a^2-b^2}}{b}$ or $\frac{-a-\sqrt{a^2-b^2}}{b}$ are the simple poles of f(z)

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$, these the roots of the equation (a)

Now the product of the roots $\alpha\beta = \frac{1}{1} = 1$

Now $|\alpha\beta| = 1 \Rightarrow |\alpha||\beta| = 1$

Since a > b > 0, $|\beta| = \left|\frac{-a-\sqrt{a^2-b^2}}{b}\right| = \left|\frac{a+\sqrt{a^2-b^2}}{b}\right|$

Here a > b ⇒ $a^2 > b^2 \Rightarrow a^2 - b^2 > 0 \Rightarrow \sqrt{a^2-b^2} > 0 \Rightarrow a + \sqrt{a^2-b^2} > a > b$

$\Rightarrow a + \sqrt{a^2-b^2} > b \Rightarrow \frac{a+\sqrt{a^2-b^2}}{b} > 1 \Rightarrow \left|\frac{a+\sqrt{a^2-b^2}}{b}\right| > 1$

Hence $z = \alpha = \frac{-a+\sqrt{a^2-b^2}}{b} < 1$ is the only simple pole lie inside the circle |z| = 1

There are no poles on the real axis

To find the residue of f(z):

Res [f,α] = $\lim_{z \to \alpha}(z-\alpha)\, f(z)$

Here f(z) = $\frac{2}{bi(z^2+\frac{2az}{b}+1)}$ and α,β are the factors of $z^2+\frac{2az}{b}+1$

⇒ f(z) = $\frac{2}{bi(z^2+\frac{2az}{b}+1)} = \frac{2}{bi(z-\alpha)(z-\beta)}$

∴ Res [f,α] = $\lim_{z \to \alpha}(z-\alpha)\, f(z) = \lim_{z \to \alpha}(z-\alpha)\frac{2}{bi(z-\alpha)(z-\beta)} = \lim_{z \to \alpha}\frac{2}{bi(z-\beta)} = \frac{2}{bi(\alpha-\beta)}$

$= \frac{2}{bi\left(\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)\right)} = \frac{2}{\frac{bi}{b}\left(-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}\right)} = \frac{2}{2i\sqrt{a^2-b^2}} = \frac{1}{i\sqrt{a^2-b^2}}$

Hence ∑ Res[f,z_k] = Res[f,α]= $\frac{1}{i\sqrt{a^2-b^2}}$

∴$\int_C f(z)dz$ = 2πi∑ Res[f,z_k] = $\frac{2\pi i}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$

∴ $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

Example I.3.

Prove that $\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2\pi}{1-a^2}$, 0 ≤ a < 1.

Solution

Let I = $\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta}$

Let z = $e^{i\theta}$ ⇒ dθ = $\frac{dz}{iz}$ and let C denotes the unit circle |z| = 1

Since z = $e^{i\theta}$ = cosθ + isinθ and $\frac{1}{z}$ = cosθ – isinθ, we have cosθ = $\frac{1}{2}$ (z + $\frac{1}{z}$)

∴ I = $\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta} = \int_C\frac{\frac{dz}{iz}}{1+a^2-2a\left(\frac{1}{2}\right)\left(z+\frac{1}{z}\right)} = \frac{1}{i}\int_C\frac{\frac{dz}{z}}{(1+a^2)-\frac{a(z^2+1)}{z}} = \frac{1}{i}\int_C\frac{\frac{dz}{z}}{\frac{(1+a^2)z-a(z^2+1)}{z}}$

$= \frac{1}{i}\int_C\frac{dz}{(1+a^2)z-a(z^2+1)} = \frac{1}{i}\int_C\frac{dz}{z+a^2z-az^2-a} = \frac{-1}{ai}\int_C\frac{dz}{\frac{-z}{a}-az+z^2+1} = \frac{-1}{ai}\int_C\frac{dz}{\left(\frac{-z}{a}+z^2\right)+(-az+1)}$

$= \frac{-1}{ai}\int_C\frac{dz}{z\left(\frac{-1}{a}+z\right)-a\left(z-\frac{1}{a}\right)} = \frac{-1}{ai}\int_C\frac{dz}{\left(\frac{-1}{a}+z\right)(z-a)} = \int_C f(z)dz$

Where f(z) = $\frac{-1}{ai\left(\frac{-1}{a}+z\right)(z-a)}$

By Cauchy's residue theorem, $\int_C f(z)dz$ = 2πi∑ Res[f,z_k] where z_k are the singularities(poles) of f(z).

To find the poles of f(z):

Poles of f(z) = zeros of $ai(-\frac{1}{a}+z)(z-a)$ and

these zeros are given by $ai(-\frac{1}{a}+z)(z-a) = 0$

⇒ $(-\frac{1}{a}+z) = 0$ or $(z-a) = 0$

⇒ z = $\frac{1}{a}$ or z = a which are the simple poles of f(z)

Since 0≤ a < 1, $\frac{1}{a} > 1$

Hence a<1 is the only pole lie inside the unit circle |z| =1

There are no poles on the real axis

To find the residue of f(z):

Res[f,a] = $\lim_{z \to a}(z-a)\, f(z) = \lim_{z \to a}(z-a)\frac{-1}{ai\left(\frac{-1}{a}+z\right)(z-a)}$

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$= \lim_{z \to a} \frac{-1}{ai\left(\frac{-1}{a}+z\right)} = \frac{-1}{ai\left(\frac{-1}{a}+a\right)} = \frac{-1}{i(-1+a^2)} = \frac{1}{i(a^2-1)}$

Hence $\sum$ Res[f,z_k] = Res[f,a] = $\frac{1}{i(a^2-1)}$

$\therefore \int_C f(z)dz = 2\pi i \sum$ Res[f,z_k] = $\frac{2\pi i}{i(a^2-1)} = \frac{2\pi}{(a^2-1)}$

$\Rightarrow$ I = $\frac{2\pi}{(a^2-1)}$

$\Rightarrow \int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2\pi}{(a^2-1)}$

Example I.4.

Evaluate $\int_0^{\pi} \frac{a\, d\theta}{a^2+\sin^2\theta}$

Solution.

First we have to change the limits to 0 to 2π from 0 to π for the given integral.

Let I = $\int_0^{\pi} \frac{a\, d\theta}{a^2+\sin^2\theta} = \int_0^{\pi} \frac{a\, d\theta}{a^2+\frac{1-\cos 2\theta}{2}} = \int_0^{\pi} \frac{2a\, d\theta}{2a^2+1-\cos 2\theta}$

To change the limit , take 2θ = ϕ ⇒ 2dθ = dϕ

If θ =0 , then ϕ = 0

If θ= π, then ϕ = 2π

Hence I = $\int_0^{2\pi} \frac{a\, d\phi}{2a^2+1-\cos\phi}$

Put z = $e^{i\phi}$ ⇒ dϕ = $\frac{dz}{iz}$ and let C denotes the unit circle |z| = 1

Since z = $e^{i\phi}$ = cosϕ + isinϕ and $\frac{1}{z}$ = cosϕ – isinϕ, we have cosϕ = $\frac{1}{2}\left(z + \frac{1}{z}\right)$

$\therefore$ I = $\int_0^{2\pi} \frac{a\, d\phi}{2a^2+1-\cos\phi} = \int_C \frac{a\frac{dz}{iz}}{2a^2+1-\frac{1}{2}\left(z+\frac{1}{z}\right)} = \int_C \frac{a\frac{dz}{iz}}{\frac{4za^2+2z-(z^2+1)}{2z}}$

$= \frac{2a}{i}\int_C \frac{dz}{4za^2+2z-z^2-1} = \frac{-2a}{i}\int_C \frac{dz}{z^2-2z(2a^2+1)+1} = \int_C f(z)dz$

Where f(z) = $\frac{-2a}{i(z^2-2z(2a^2+1)+1)}$

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum$ Res[f,z_k] where z_k are the singularities(poles) of f(z).

To find the poles of f(z):

Poles of f (z) = zeros of $i(z^2-2z(2a^2+1)+1)$, these zeros are given by $i(z^2-2z(2a^2+1)+1) = 0$

$\Rightarrow z^2-2z(2a^2+1)+1 = 0$

$\Rightarrow z = \frac{2(2a^2+1)\pm\sqrt{(2(2a^2+1))^2-4(1)(1)}}{2(1)} = \frac{2(2a^2+1)\pm 2\sqrt{((2a^2+1))^2-1}}{2}$

$= 2a^2+1 \pm \sqrt{(2a^2+1)^2-1} = 2a^2+1 \pm \sqrt{4a^4+1+4a^2-1}$

$= 2a^2+1 \pm \sqrt{4a^4+4a^2} = 2a^2+1 \pm 2a\sqrt{a^2+1}$

$\therefore$ z = $2a^2+1+2a\sqrt{a^2+1}$ = α (say)

Or z = $2a^2+1-2a\sqrt{a^2+1}$ = β (say)

Hence the poles of f(z) are α,β which are simple poles.

Now α,β are the roots of the equation $z^2-2z(2a^2+1)+1 = 0$

Product of the roots αβ = 1 ⇒ | αβ| = 1⇒|α||β| = 1

Clearly |α| = $|2a^2+1+2a\sqrt{a^2+1}|$ > 1 ⇒ |β| < 1

$\therefore$ the only pole lie inside the unit circle |z| =1 is β = $2a^2+1-2a\sqrt{a^2+1}$

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There are no poles on the real axis

To find the residue of f(z) :

Res[f,β] = $\lim_{z \to \beta}(z-\beta)\, f(z) = \lim_{z \to \beta}(z-\beta)\, \frac{-2a}{i(z^2-2z(2a^2+1)+1)}$

$= \lim_{z \to \beta}(z-\beta)\, \frac{-2a}{i(z-\alpha)(z-\beta)} = \lim_{z \to \beta} \frac{-2a}{i(z-\alpha)} = \frac{-2a}{i(\beta-\alpha)} = \frac{-2a}{i(2a^2+1-2a\sqrt{a^2+1} - \left(2a^2+1+2a\sqrt{a^2+1}\right))}$

$= \frac{-2a}{i(2a^2+1-2a\sqrt{a^2+1} -2a^2-1-2a\sqrt{a^2+1})} = \frac{-2a}{i(-4a\sqrt{a^2+1})} = \frac{1}{2i\sqrt{a^2+1})}$

Hence $\sum$ Res[f,z_k] = Res[f,β] = $\frac{1}{2i\sqrt{a^2+1})}$

$\therefore \int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k] = \frac{2\pi i}{2i\sqrt{a^2+1})} = \frac{\pi}{\sqrt{a^2+1})}$

$\Rightarrow I = \int_0^{\pi} \frac{a\, d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{a^2+1})}$

Example I.5.

Evaluate $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$ (a > 0, b > 0 ; a > b)

Solution

Let I = $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$

Take z = $e^{i\theta}$ ⇒ dθ = $\frac{dz}{iz}$ and let C denotes the unit circle |z| = 1

Since z = $e^{i\theta}$ = cosθ + isinθ and $\frac{1}{z}$ = cosθ – isinθ, we have cosθ = $\frac{1}{2}$ (z + $\frac{1}{z}$)

$\Rightarrow I = \int_C \frac{\frac{dz}{iz}}{\left(a+\frac{b}{2}(z+\frac{1}{z})\right)^2} = \int_C \frac{\frac{dz}{iz}}{\left(\frac{2az+b(z^2+1)}{2z}\right)^2} = \frac{4}{i}\int_C \frac{zdz}{(2az+b(z^2+1))^2} = \frac{4}{i}\int_C \frac{zdz}{(2az+bz^2+b))^2}$

$= \int_C f(z)dz$ where f(z) = $\frac{4z}{i(2az+bz^2+b))^2}$

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum$ Res[f,z_k] where z_k are the singularities(poles) of f(z).

To find the poles of f(z):

Poles of f(z) = zeros of $i(2az+bz^2+b)^2$, these zeros are given by $i(2az+bz^2+b)^2 = 0$

$\Rightarrow bz^2+2az+b = 0 \Rightarrow z^2+\frac{2az}{b}+1 = 0$

$\Rightarrow z = \frac{-\frac{2a}{b}\pm\sqrt{(\frac{2a}{b})^2-4}}{2} = \frac{-2a\pm\sqrt{4a^2-4b^2}}{2b} = \frac{-a\pm\sqrt{a^2-b^2}}{b}$

$\Rightarrow z = \frac{-a+\sqrt{a^2-b^2}}{b}$ = β (say)

Or z = $\frac{-a-\sqrt{a^2-b^2}}{b}$ = α (say)

Hence the poles of f(z) are α,β both order 2

Since α,β are the roots of the equation $bz^2+2az+b = 0$

Product of the root αβ = b/b = 1 ⇒ | αβ| = 1 ⇒ |α||β| = 1

Given a > b ⇒ $a^2 > b^2 \Rightarrow a^2- b^2>0 \Rightarrow \sqrt{a^2-b^2} > 0 \Rightarrow a + \sqrt{a^2-b^2} > a > b$

$\Rightarrow a + \sqrt{a^2-b^2} > b \Rightarrow \frac{a+\sqrt{a^2-b^2}}{b} > 1 \Rightarrow \left|\frac{a+\sqrt{a^2-b^2}}{b}\right| > 1$

Hence z = β = $\frac{-a+\sqrt{a^2-b^2}}{b}$ < 1 is the only pole lie inside the circle |z| = 1

There are no poles on the real axis

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To fine the residue of f(z):

$$\text{Res}\,[f,\beta] = \lim_{z\to\beta}\frac{d}{dz}(z-\beta)^2 f(z) = \lim_{z\to\beta}\frac{d}{dz}(z-\beta)^2\frac{4z}{ib^2\left(\frac{2az}{b}+z^2+1\right)^2}$$

$$= \lim_{z\to\beta}\frac{d}{dz}(z-\beta)^2\frac{4z}{ib^2((z-\alpha)(z-\beta))^2} = \lim_{z\to\beta}\frac{d}{dz}(z-\beta)^2\frac{4z}{i\,b^2(z-\alpha)^2(z-\beta)^2}$$

$$= \lim_{z\to\beta}\frac{d}{dz}\frac{4z}{i\,b^2(z-\alpha)^2} = \lim_{z\to\beta}\frac{4}{ib^2}\frac{d}{dz}\frac{z}{(z-\alpha)^2} = \lim_{z\to\beta}\frac{4}{ib^2}\left[\frac{(z-\alpha)^2-z2(z-\alpha)}{(z-\alpha)^4}\right] =$$

$$= \lim_{z\to\beta}\frac{4}{ib^2}(z-\alpha)\left[\frac{(z-\alpha)-z2}{(z-\alpha)^4}\right] = \lim_{z\to\beta}\frac{4}{ib^2}\left[\frac{(z-\alpha)-z2}{(z-\alpha)^3}\right] = \frac{4}{ib^2}\left[\frac{(\beta-\alpha)-2\beta}{(\beta-\alpha)^3}\right]$$

$$= \frac{-4}{ib^2}\left[\frac{\alpha+\beta}{(\beta-\alpha)^3}\right] = \frac{-4}{ib^2}\left[\frac{\alpha+\beta}{(\beta-\alpha)^3}\right] = \frac{-4}{ib^2}\left[\frac{\frac{-a-\sqrt{a^2-b^2}}{b}+\frac{-a+\sqrt{a^2-b^2}}{b}}{\left(\frac{-a+\sqrt{a^2-b^2}}{b}-\frac{-a-\sqrt{a^2-b^2}}{b}\right)^3}\right] = \frac{-4}{ib^2}\left[\frac{\frac{-2a}{b}}{\left(\frac{-a+\sqrt{a^2-b^2}}{b}+\frac{a+\sqrt{a^2-b^2}}{b}\right)^3}\right]$$

$$= \frac{-4}{ib^2}\left[\frac{\frac{-2a}{b}}{\left(\frac{2\sqrt{a^2-b^2}}{b}\right)^3}\right] = \left[\frac{8ab^3}{8b^3 i\left(\sqrt{a^2-b^2}\right)^3}\right] = \frac{a}{i\left(\sqrt{a^2-b^2}\right)^3}$$

Hence $\sum \text{Res}[f,z_k] = \text{Res}[f,\beta] = \dfrac{a}{i\left(\sqrt{a^2-b^2}\right)^3}$

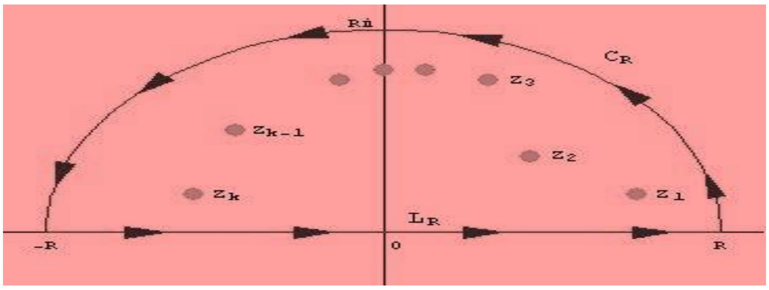
$$\therefore \int_C f(z)dz = 2\pi i\sum \text{Res}[f,z_k] = \frac{2\pi i a}{i\left(\sqrt{a^2-b^2}\right)^3} = \frac{2\pi a}{\left(\sqrt{a^2-b^2}\right)^3}$$

$$\Rightarrow I = \int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{\left(\sqrt{a^2-b^2}\right)^3}$$

Type II.

Evaluation of the integral $\int_{-\infty}^{\infty} f(x)dx$ where f(x) is a real rational function of the real variable x

If the rational function f(x) = $\frac{g(x)}{h(x)}$, then degree of h(x) exceeds that of g(x) and g(x) ≠ 0.To find the value of the integral, by inventing a closed contour in the complex plane which includes the required integral. For this we have to close the contour by a very large semi-circle in the upper half-plane. Suppose we use the symbol "*R*" for the radius. The entire contour integral comprises the integral along the real axis from $-R$ to $+R$ together with the integral along the semi-circular arc. In the limit as $R\to\infty$the contribution from the straight line part approaches the required integral, while the curved section may in some cases vanish in the limit.



The poles $z_1, z_2, \ldots, z_k$ of $\frac{g(x)}{h(x)}$, that lie in the upper half-plane

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty}\frac{g(x)}{h(x)}dx = 2\pi i\sum \text{Res}[f,z_k]$$

Example II.1

Using the residue of calculus compute $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)}$

Solution

Consider the integral $\int_C f(z)dz$ where f(z) = $\frac{1}{(z^2+1)(z^2+4)}$

To find the poles of f(z) :

The poles of f(z) = zeros of $(z^2+1)(z^2+4)$, these zeros are given by $(z^2+1)(z^2+4) = 0$

$\Rightarrow z^2+1 = 0$ or $z^2+4 = 0$

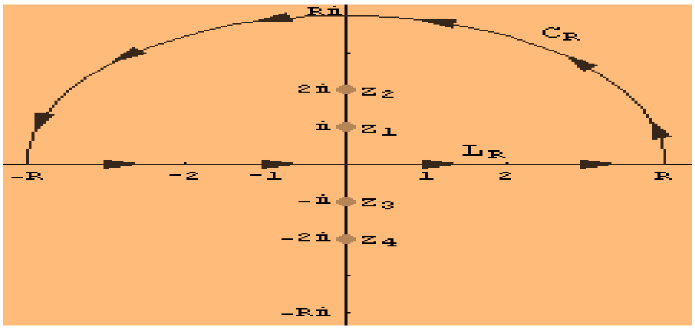
$\Rightarrow z^2 = -1 = i^2 \Rightarrow z = \pm i$

Or $z^2 = -4 = (2i)^2 \Rightarrow z = \pm 2i$

Hence the poles of f(z) are ± i, ± 2i (all are simple poles)

And the poles z = i and z = 2i are the only poles lie inside the upper half of semi-circle.

There are no poles on the real axis



By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ ------------- (a) (on the real line –R to R (L_R)+ the upper half of the semi circle C_R)

To find the residue of f(z):

$$\text{Res}[f,i] = \lim_{z \to i}(z-i)\ f(z) = \lim_{z \to i}(z-i)\ \frac{1}{(z^2+1)(z^2+4)} = = \lim_{z \to i}(z-i)\ \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

$$= \lim_{z \to i} \frac{1}{(z+i)(z+2i)(z-2i)} = \frac{1}{(i+i)(i+2i)(i-2i)} = \frac{1}{(2i)(3i)(-i)} = \frac{1}{6i} = -\frac{i}{6}$$

$$\text{Res}[f,2i] = \lim_{z \to 2i}(z-2i)\ f(z) = \lim_{z \to 2i}(z-2i)\ \frac{1}{(z^2+1)(z^2+4)} = = \lim_{z \to 2i}(z-2i)\ \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

$$= \lim_{z \to 2i} \frac{1}{(z-i)(z+i)(z+2i)} = \frac{1}{(2i-i)(i+2i)(2i+2i)} = \frac{1}{(i)(3i)(4i)} = \frac{-1}{12i} = \frac{i}{12}$$

Consider $\left|\int_{C_R} f(z)dz\right| = \left|\int_{C_R} \frac{1}{(z^2+1)(z^2+4)} dz\right| \le \int_{C_R} \left|\frac{dz}{(z^2+1)(z^2+4)}\right| \le \int_{C_R} \frac{|dz|}{|(z^2+1)(z^2+4)|}$

$\le \int_{C_R} \frac{|dz|}{(|z|^2-1)(|z|^2-4)}$ ---------(b)

Let $z = Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$

$\Rightarrow |dz| = |iRe^{i\theta}d\theta| = R\,d\theta$ ($\because |i| = 1 = |e^{i\theta}|$)

If z = -R, then $Re^{i\theta} = -R \Rightarrow e^{i\theta} = -1 \Rightarrow \theta = \pi$

If z = R, then $Re^{i\theta} = R \Rightarrow e^{i\theta} = 1 \Rightarrow \theta = 0$

Hence (b) $\Rightarrow \left|\int_{C_R} f(z)dz\right| \le \int_0^{\pi} \frac{Rd\theta}{(R^2-1)(R^2-4)} = \frac{R}{(R^2-1)(R^2-4)} \int_0^{\pi} d\theta = \frac{R\pi}{(R^2-1)(R^2-4)}$

As $R \to \infty$, $\frac{R\pi}{(R^2-1)(R^2-4)} \to 0 \Rightarrow \int_{C_R} f(z)dz \to 0$

Now as $R \to \infty$, (a) $\Rightarrow \int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + 0 = \int_{-\infty}^{\infty} f(x)dx$

Where $f(x) = \frac{1}{(x^2+1)(x^2+4)}$

Hence $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx = \int_{-\infty}^{\infty} f(x)dx = \int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k]$

$$= 2\pi i\{\text{Res}[f,i] + \text{Res}[f,2i]\} = 2\pi i\{\frac{-i}{6} + \frac{i}{12}\} = 2\pi i(\frac{-i}{12}) = \frac{\pi}{6}$$

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Example II.2

Using the residue of calculus compute $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$

Solution

Consider the integral $\int_C f(z)dz$ where f(z) = $\frac{1}{(z^2+4)^3}$

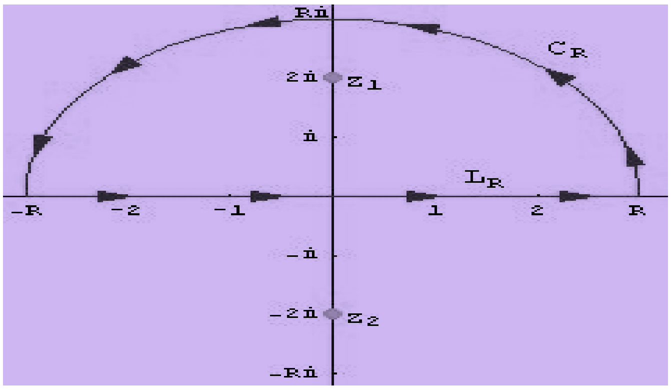
To find the poles of f(z):

Poles of f(z) = zeros of $(z^2+4)^3$, these zeros are given by $(z^2+4)^3 = 0$

$\Rightarrow z^2 +4 = 0 \Rightarrow z^2 = -4 \Rightarrow z^2 = (2i)^2$

$\Rightarrow z = \pm 2i \Rightarrow z = 2i$ or $z = -2i$

Hence the poles of f(z) are z = 2i , z = -2i , both of order 3



The only pole lie inside the upper half of the semi-circle is z = 2i of order 3

There are no poles on the real axis

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ (on the real line –R to R (L_R)+ the upper half of the semi circle C_R)

To find the residue of f(z):

$$\text{Res}[f,2i] = \frac{1}{(2)!} \lim_{z \to 2i} \frac{d^2}{dz^2}(z-2i)^3 f(z) = \frac{1}{2} \lim_{z \to 2i} \frac{d^2}{dz^2}(z-2i)^3 \frac{1}{(z^2+4)^3}$$

$$= \frac{1}{2} \lim_{z \to 2i} \frac{d^2}{dz^2}(z-2i)^3 \frac{1}{((z-2i)(z+2i))^3} = \frac{1}{2} \lim_{z \to 2i} \frac{d^2}{dz^2}(z-2i)^3 \frac{1}{(z-2i)^3(z+2i)^3}$$

$$= \frac{1}{2} \lim_{z \to 2i} \frac{d^2}{dz^2} \frac{1}{(z+2i)^3} = \frac{1}{2} \lim_{z \to 2i} \frac{d}{dz}\frac{d}{dz}\left(\frac{1}{(z+2i)^3}\right) = \frac{1}{2} \lim_{z \to 2i} \frac{d}{dz}\left(\frac{-3(z+2i)^2}{(z+2i)^6}\right)$$

$$= \frac{1}{2} \lim_{z \to 2i} \frac{d}{dz}\left(\frac{-3}{(z+2i)^4}\right) = \frac{1}{2} \lim_{z \to 2i} \left(\frac{3(4)(z+2i)^3}{(z+2i)^8}\right) = \frac{12}{2} \lim_{z \to 2i} \left(\frac{1}{(z+2i)^5}\right)$$

$$= 6\left(\frac{1}{(2i+2i)^5}\right) = \frac{6}{(4i)^5} = \frac{6}{1024i} = \frac{3}{512i}$$

Consider $\left|\int_{C_R} f(z)dz\right| = \left|\int_{C_R} \frac{1}{(z^2+4)^3} dz\right| \leq \int_{C_R} \left|\frac{dz}{(z^2+4)^3}\right| \leq \int_{C_R} \frac{|dz|}{|(z^2+4)^3|}$

$\leq \int_{C_R} \frac{|dz|}{(|z|^2-4)^3}$ ---------(b)

Let $z = Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$

$\Rightarrow |dz| = |iRe^{i\theta}d\theta| = R\, d\theta$ ($\because |i| = 1 = |e^{i\theta}|$)

If z = -R, then $Re^{i\theta} = -R \Rightarrow e^{i\theta} = -1 \Rightarrow \theta = \pi$

If z = R, then $Re^{i\theta} = R \Rightarrow e^{i\theta} = 1 \Rightarrow \theta = 0$

Hence (b) $\Rightarrow \left|\int_{C_R} f(z)dz\right| \leq \int_0^{\pi} \frac{R d\theta}{(R^2-4)^3} = \frac{R}{(R^2-4)^3} \int_0^{\pi} d\theta = \frac{R\pi}{(R^2-4)^3}$

As R $\to \infty$, $\frac{R\pi}{(R^2-4)^3} \to 0 \Rightarrow \int_{C_R} f(z)dz \to 0$

Now as R$\to \infty$, (a) $\Rightarrow \int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + 0 = \int_{-\infty}^{\infty} f(x)dx$

Where f(x) = $\frac{1}{(x^2+4)^3}$

Hence $\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3} dx = \int_{-\infty}^{\infty} f(x)dx = \int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k]$

$= 2\pi i\{\text{Res}[f,2i]\} = 2\pi i\{\frac{3}{i512}\} = \frac{3\pi}{256}$

Example II.3

Prove that $\int_{-\infty}^{\infty} \frac{(x^2-x+2)\,dx}{(x^4+10x^2+9)} = \frac{5\pi}{12}$

Solution

Consider the integral $\int_C f(z)dz$ where f(z) = $\frac{(z^2-z+2)}{(z^4+10z^2+9)}$

To find the poles of f(z):

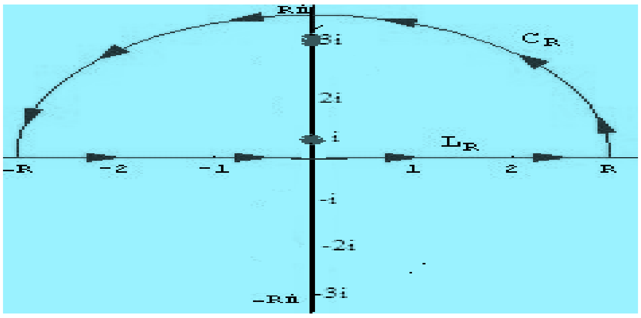
Poles of f(z) = zeros of z^4+10z^2+9 , these zeros are given by $z^4+10z^2+9 = 0$

$\Rightarrow z^4+z^2+9z^2+9=0 \Rightarrow (z^2+1)(z^2+9) = 0$

$\Rightarrow z^2 = -1 = i^2$ or $z^2 = -9 = (3i)^2$

$\Rightarrow z = \pm i$ or $z = \pm 3i$

Hence the poles of f(z) are i,-i,3i,-3i (all are simple poles)



The poles that are lying the upper half of the semi-circle are i,3i

There are no poles on the real axis

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ (on the real line –R to R (L_R)+ the upper half of the semi-circle C_R)

To find the residue of f(z):

Res[f,i] = $\lim_{z \to i}(z-i)\ f(z) = \lim_{z \to i}(z-i)\ \frac{(z^2-z+2)}{(z^4+10z^2+9)} = \lim_{z \to i}(z-i)\ \frac{(z^2-z+2)}{(z-i)(z+i)(z+3i)(z-3i)}$

$= \lim_{z \to i} \frac{(z^2-z+2)}{(z+i)(z+3i)(z-3i)} = \frac{(i^2-i+2)}{(i+i)(i+3i)(i-3i)} = \frac{1-i}{(2i)(4i)(-2i)} = \frac{1-i}{16i}$

Res[f,3i] = $\lim_{z \to 3i}(z-3i)\ f(z) = \lim_{z \to 3i}(z-3i)\ \frac{(z^2-z+2)}{(z^4+10z^2+9)}$

$= \lim_{z \to 3i}(z-3i)\ \frac{(z^2-z+2)}{(z-i)(z+i)(z+3i)(z-3i)} = \lim_{z \to 3i} \frac{(z^2-z+2)}{(z-i)(z+i)(z+3i)} = \frac{((3i)^2-(3i)+2)}{(3i-i)(3i+i)(3i+3i)}$

$= \frac{-7-3i}{(2i)(4i)(6i)} = \frac{-(7+3i)}{-48i} = \frac{7+3i}{48i}$

Consider $\left|\int_{C_R} f(z)dz\right| = \left|\int_{C_R} \frac{(z^2-z+2)}{(z^4+10z^2+9)} dz\right| \le \int_{C_R} \left|\frac{(z^2-z+2)\,dz}{(z^4+10z^2+9)}\right| \le \int_{C_R} \frac{|(z^2-z+2)||\,dz|}{|(z^4+10z^2+9)|}$

$\le \int_{C_R} \frac{|(z^2-z+2)||\,dz|}{|(z^2+1)||(z^2+9)|} \le \int_{C_R} \frac{(|z|^2-|z|+2)|dz|}{(|z|^2-1)(|z|^2-9)}$ ---------(b)

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Let $z = Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$

$\Rightarrow |dz| = |iRe^{i\theta}d\theta| = R\, d\theta$ ($\because |i| = 1 = |e^{i\theta}|$)

If $z = -R$, then $Re^{i\theta} = -R \Rightarrow e^{i\theta} = -1 \Rightarrow \theta = \pi$

If $z = R$, then $Re^{i\theta} = R \Rightarrow e^{i\theta} = 1 \Rightarrow \theta = 0$

Hence (b) $\Rightarrow \left|\int_{C_R} f(z)dz\right| \leq \int_0^\pi \frac{R^2 d\theta}{(R^2-1)(R^2-9)} - \int_0^\pi \frac{R d\theta}{(R^2-1)(R^2-9)} + \int_0^\pi \frac{2 d\theta}{(R^2-1)(R^2-9)} = \frac{R^2}{(R^2-1)(R^2-9)}\int_0^\pi d\theta - \frac{R}{(R^2-1)(R^2-9)}\int_0^\pi d\theta + \frac{2}{(R^2-1)(R^2-9)}\int_0^\pi d\theta$

$= \frac{R^2\pi}{(R^2-1)(R^2-9)} - \frac{R\pi}{(R^2-1)(R^2-9)} + \frac{2\pi}{(R^2-1)(R^2-9)}$

As $R \to \infty$, $\frac{R^2\pi}{(R^2-1)(R^2-9)} \to 0$, $\frac{R\pi}{(R^2-1)(R^2-9)} \to 0$ and $\frac{2\pi}{(R^2-1)(R^2-9)} \to 0 \quad \Rightarrow \int_{C_R} f(z)dz \to 0$

Now as $R \to \infty$, (a) $\Rightarrow \int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + 0 = \int_{-\infty}^{\infty} f(x)dx$

Where $f(x) = \frac{(x^2-x+2)}{(x^4+10x^2+9)}$

Hence $\int_{-\infty}^{\infty} \frac{(x^2-x+2)}{(x^4+10x^2+9)}dx = \int_{-\infty}^{\infty} f(x)dx = \int_C f(z)dz = 2\pi i \sum \text{Res}[f, z_k]$

$= 2\pi i \left\{\frac{1-i}{16i} + \frac{7+3i}{48i}\right\} = 2\pi i \left\{\frac{3-3i+7+3i}{48i}\right\} = \frac{10\pi}{24} = \frac{5\pi}{12}$

Example II.4

Evaluate $\int_0^\infty \frac{dx}{x^4+a^4}$

Solution

Let us take $\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4}$

Consider the integral $\int_C f(z)dz$ where $f(z) = \frac{1}{z^4+a^4}$

To find the poles of f(z):

Poles of f(z) = zeros of z^4+a^4 , these zeros are given by $z^4+a^4 = 0$

$\Rightarrow z^4 = -a^4 \Rightarrow z^4 = a^4e^{i\pi}$ ($\because e^{i\pi} = -1$)

$\Rightarrow z^4 = a^4e^{i\pi}e^{i2n\pi}$ ($\because e^{i2n\pi} = 1$)

$\Rightarrow z^4 = a^4e^{i\pi+2n\pi i} = a^4e^{i(2n+1)\pi}$

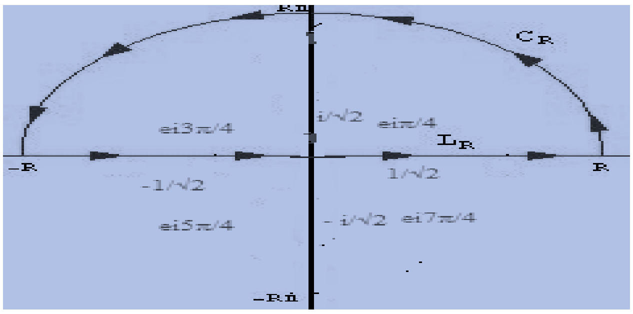
$\Rightarrow z = a\, e^{i(2n+1)\pi/4}$, n = 0,1,2,3

If n=0, $z = a\, e^{i\pi/4} = a(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = a(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = \alpha$ (say)

If n = 1, $z = a\, e^{i3\pi/4} = a(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = a(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = \beta$ (say)

If n = 2, $z = a\, e^{i5\pi/4} = a(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}) = -a(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = \gamma$(say)

If n=3, $z = a\, e^{i7\pi/4} = a(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}) = a(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) = \delta$(say)



The poles lying inside the upper hemi circle are $ae^{i\pi/4} = \alpha$, $ae^{i3\pi/4} = \beta$ (both are simple poles)

There are no poles on the real axis

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i\sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ (on the real line –R to R (L_R)+ the upper half of the semi-circle C_R)

To find the residue of f(z):

$\text{Res}[f,\alpha] = \lim_{z\to\alpha}(z-\alpha)\, f(z) = \lim_{z\to\alpha}(z-\alpha)\,\frac{1}{z^4+a^4}$

It is difficult to solve while factoring $\frac{1}{z^4+a^4}$ and taking limit, so we will use L'Hospital rule (that is differentiating Nr and Dr separately w.r.to z)

$= \lim_{z\to\alpha}\frac{1}{4z^3} = \frac{1}{4\alpha^3} = \frac{\alpha}{4\alpha^4}$

Now $\alpha = a\, e^{i\pi/4} \Rightarrow \alpha^4 = a^4e^{i\pi} \Rightarrow \alpha^4 = -a^4$ ($\because e^{i\pi} = -1$)

$\therefore$ Res $[f,\alpha] = -\frac{\alpha}{4a^4} = -\frac{ae^{\frac{i\pi}{4}}}{4a^4} = -\frac{e^{\frac{i\pi}{4}}}{4a^3}$

Now $\beta = a\, e^{i3\pi/4} \Rightarrow \beta^4 = a^4e^{i3\pi} \Rightarrow \beta^4 = -a^4$ ($\because e^{i3\pi} = -1$)

Similarly , Res $[f,\beta] = \frac{1}{4\beta^3} = \frac{\beta}{4\beta^4} = -\frac{ae^{\frac{3i\pi}{4}}}{4a^4} = -\frac{e^{\frac{3i\pi}{4}}}{4a^3}$

Consider $\left|\int_{C_R} f(z)dz\right| = \left|\int_{C_R}\frac{1}{z^4+a^4}dz\right| \le \int_{C_R}\left|\frac{dz}{z^4+a^4}\right| \le \int_{C_R}\frac{|dz|}{|z^4+a^4|}$

$\le \int_{C_R}\frac{|dz|}{|z|^4-a^4}$ ---------(b)

Let $z = Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$

$\Rightarrow |dz| = |iRe^{i\theta}d\theta| = R\, d\theta$ ($\because |i| = 1 = |e^{i\theta}|$)

If z = -R, then $Re^{i\theta} = -R \Rightarrow e^{i\theta} = -1 \Rightarrow \theta = \pi$

If z = R, then $Re^{i\theta} = R \Rightarrow e^{i\theta} = 1 \Rightarrow \theta = 0$

Hence (b) $\Rightarrow \left|\int_{C_R} f(z)dz\right| \le \int_0^{\pi}\frac{Rd\theta}{(R^4-a^4)} \le \frac{R}{(R^4-a^4)}\int_0^{\pi} d\theta \le \frac{R\pi}{(R^4-a^4)}$

As $R\to\infty$, $\frac{R\pi}{(R^4-a^4)} \to 0 \Rightarrow \int_{C_R} f(z)dz \to 0$

Now as $R\to\infty$, (a) $\Rightarrow \int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + 0 = \int_{-\infty}^{\infty} f(x)dx$

Where $f(x) = \frac{1}{(x^4+a^4)}$

Hence $\int_{-\infty}^{\infty}\frac{1}{(x^4+a^4)}dx = \int_{-\infty}^{\infty} f(x)dx = \int_C f(z)dz = 2\pi i\sum \text{Res}[f,z_k]$

$= 2\pi i\left\{-\frac{e^{\frac{i\pi}{4}}}{4a^3} - \frac{e^{\frac{i3\pi}{4}}}{4a^3}\right\} = -\frac{2\pi i}{4a^3}\left\{e^{\frac{i\pi}{4}} + e^{\frac{i3\pi}{4}}\right\} = -\frac{\pi i}{2a^3}\left\{\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) + \left(\frac{-1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right\}$

$= \frac{2\pi}{2\sqrt{2}a^3} = \frac{\pi}{\sqrt{2}a^3}$

We know that $\int_{-\infty}^{\infty}\frac{1}{(x^4+a^4)}dx = 2\int_0^{\infty}\frac{1}{(x^4+a^4)}dx$

$\Rightarrow \int_0^{\infty}\frac{1}{(x^4+a^4)}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{(x^4+a^4)}dx = \frac{\pi}{2\sqrt{2}a^3}$

Type III.

Evaluation of the integral $\int_{-\infty}^{\infty} f(x)\sin mx\, dx$, $\int_{-\infty}^{\infty} f(x)\cos mx\, dx$ where m > 0 and f(x) is a real rational function of the real variable x

If the rational function $f(x) = \frac{g(x)}{h(x)}$, then degree of h(x) exceeds that of g(x) and $g(x) \neq 0$.

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Let g(x) and h(x) be polynomials with real coefficients, of degree p and q, respectively, where q $\geq$ p+1.

If h(x) $\neq 0$ for all real x, and m is a real number satisfying m > 0, then

$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos\ mx\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{g(x)}{h(x)} \cos mx\, dx$ and $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} sinmx\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{g(x)}{h(x)} \sin mx\, dx$

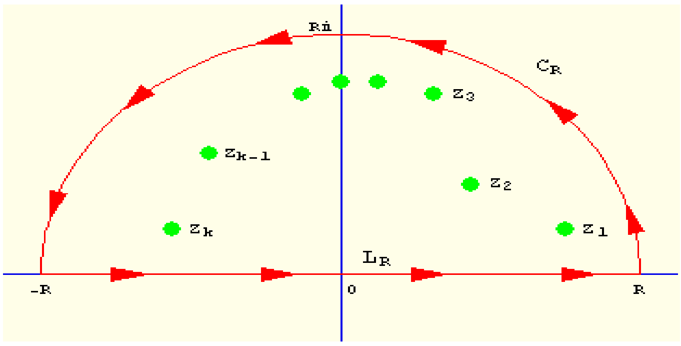
We know that Euler's formula e^{imx} = cos mx + i sin mx , where cos mx = Re[e^{imx}]

and sin mx = Im[e^{imx}] , m is a positive real.

We have $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{imx} dx = \int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos\ mx\, dx$ + i $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} sinmx\, dx$

Here we are going to use the complex function f(z) = $\frac{g(z)}{h(z)} e^{imz}$ to evaluate the given integral. $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos\ mx\, dx$ = Re {2πi ∑ Res[f,z_k]} and

$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} sinmx\, dx$ = Im {2πi ∑ Res[f,z_k]}, where $z_1, z_2, \ldots, z_k$ are the poles lies on the upper half of the semi-circle.



Lemma III.1.(Jordan's Lemma)

If f (z) ⟶ 0 uniformly as z ⟶∞, then $\lim_{R\to\infty} \int_{C_1} e^{imz} f(z) dz$ = 0, (m > 0) where C_1 denotes the semi-circle |z| = R, I(z) > 0.

Proof.

Given f (z) ⟶ 0 uniformly as z ⟶∞

⇒ given $\varepsilon > 0$, ∃ a $R_0 > 0$ such that | f(z) – 0| < ε , ∀ R ≥ R_0

That is | f(z)| < ε , ∀ R ≥ R_0 --------(a)

Let |z| = R which is the semi-circle

Put z = $Re^{i\theta}$ ⇒ dz = R $e^{i\theta}$ i dθ ⇒ dz = izdθ, 0≤ θ ≤ π

Now $e^{imz} = e^{ime^{i\theta}} = e^{imR(cos\theta + isin\theta)} = e^{imRcos\theta - mRsin\theta} = e^{imRcos\theta}\ e^{-mRsin\theta}$

⇒ $|e^{imR(cos\theta+isin\theta)}| = |e^{imRcos\theta}|\ |e^{-mRsin\theta}| = e^{-mRsin\theta}$ (∵$|e^{imRcos\theta}|$ = 1) ----- (b)

We know that $\frac{sin\theta}{\theta}$ is monotonically decreases as θ increases from 0 to $\frac{\pi}{2}$.

If $0 \leq \theta \leq \frac{\pi}{2}$, then $\frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} \leq \frac{sin\theta}{\theta}$

$\Rightarrow \frac{1}{\frac{\pi}{2}} \leq \frac{sin\theta}{\theta} \Rightarrow \frac{2}{\pi} \leq \frac{sin\theta}{\theta} \Rightarrow \sin\theta \geq \frac{2\theta}{\pi}$

$\Rightarrow -\ \frac{2\theta}{\pi} \geq -\ \sin\theta$

$\Rightarrow -\ \frac{mR2\theta}{\pi} \geq -\ mR\sin\theta$

$\Rightarrow\ e^{-\frac{mR2\theta}{\pi}} \geq e^{-mRsin\theta}$ ------ (c)

From (a), (b) and (c),

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$$\left|\int_{C_1} e^{imz} f(z)dz\right| \leq \int_{C_1} |e^{imz} f(z)dz|$$
$$\leq \int_{C_1} |e^{imz}||f(z)||dz|$$
$$\leq \varepsilon \int_0^{\pi} e^{-mR\sin\theta}\, R\, d\theta \qquad (\because |dz| = |izd\theta| = |z|d\theta = Rd\theta)$$
$$\leq 2\varepsilon R \int_0^{\frac{\pi}{2}} e^{-mR\sin\theta} d\theta$$
$$\leq 2\varepsilon R \int_0^{\frac{\pi}{2}} e^{-\frac{mR2\theta}{\pi}} d\theta$$
$$\leq 2\varepsilon R \left[\frac{e^{-\frac{mR2\theta}{\pi}}}{-\frac{2mR}{\pi}}\right]_0^{\frac{\pi}{2}}$$
$$\leq 2\varepsilon R\{(e^{-mR})\frac{\pi}{-2mR} - (e^0)\frac{\pi}{-2mR}\}$$
$$\leq \frac{2\varepsilon R\pi}{2mR}\{(-e^{-mR} + 1\}$$
$$\leq \frac{\varepsilon\pi}{m}\{-e^{-mR} + 1\} \leq \frac{\varepsilon\pi}{m} \qquad (\because \{-e^{-mR} + 1\} < 1)$$
$$\left|\int_{C_1} e^{imz} f(z)dz\right| \leq \frac{\varepsilon\pi}{m} = \varepsilon' \text{ (say)}$$

As lim $R \to \infty$, $\left|\int_{C_1} e^{imz} f(z)dz - 0\right| \leq \varepsilon'$

$\Rightarrow \int_{C_1} e^{imz} f(z)dz \to 0$ as $R \to \infty$

Example III.1.

Use the method of contour integration to prove that $\int_0^{\infty} \frac{\cos mx}{(x^2+a^2)} dx = \frac{\pi e^{-ma}}{2a}$ and $\int_0^{\infty} \frac{\sin mx}{(x^2+a^2)} dx = 0$

Solution.

Consider the integral $\int_C f(z)dz$ where f(z) = $\frac{e^{imz}}{(z^2+a^2)}$

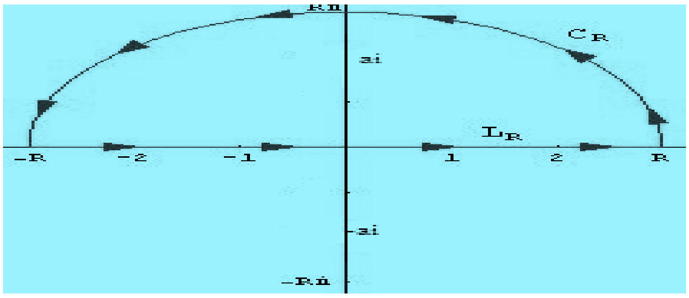
To find the poles of f(z):

Poles of f(z) = zeros of z^2+a^2 , these zeros are given by $z^2+a^2 = 0$

$\Rightarrow z^2 = -a^2 \Rightarrow z = (ai)^2 \Rightarrow z = \pm ai \Rightarrow z = ai$ or $z = -ai$

Poles of f (z) are ai, -ai (both are simple poles)

There are no poles on the real axis



The only pole lie inside the upper half of semi-circle is z = ai

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ (on the real line –R to R (L_R)+ the upper half of the semi-circle C_R).

To find the residue of f(z):

$$\text{Res}[f,ai] = \lim_{z \to ai}(z-ai)\, f(z) = \lim_{z \to ai}(z-ai)\, \frac{e^{imz}}{(z^2+a^2)} = \lim_{z \to ai}(z-ai)\, \frac{e^{imz}}{(z-ai)(z+ai)}$$

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$= \lim_{z \to ai} \frac{e^{imz}}{(z+ai)} = \frac{e^{imai}}{(ai+ai)} = \frac{e^{-ma}}{2ai}$

Now $\lim_{z \to \infty} \frac{1}{(z^2+a^2)} = 0$

∴ by Jordan's lemma , $\lim_{R \to \infty} \int_{C_R} \frac{e^{imz}}{(z^2+a^2)} dz = 0$

As R → ∞ ,

$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + \lim_{R \to \infty} \int_{C_R} \frac{e^{imz}}{(z^2+a^2)} dz = \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)} dx + 0$

$= \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)} dx$

Hence $\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)} dx = \int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k] = 2\pi i \frac{e^{-ma}}{2ai} = \frac{\pi e^{-ma}}{a}$

$\Rightarrow \int_{-\infty}^{\infty} \frac{cosmx+isinmx}{(x^2+a^2)} dx = \frac{\pi e^{-ma}}{a}$

$\Rightarrow \int_{-\infty}^{\infty} \frac{cosmx}{(x^2+a^2)} dx + i \int_{-\infty}^{\infty} \frac{sinmx}{(x^2+a^2)} dx = \frac{\pi e^{-ma}}{a}$

Equating real and imaginary parts,

$\int_{-\infty}^{\infty} \frac{cosmx}{(x^2+a^2)} dx = \frac{\pi e^{-ma}}{a}$ and $\int_{-\infty}^{\infty} \frac{sinmx}{(x^2+a^2)} dx = 0$

$\Rightarrow \int_0^{\infty} \frac{cosmx}{(x^2+a^2)} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{cosmx}{(x^2+a^2)} dx = \frac{\pi e^{-ma}}{2a}$

Example III.2.

Apply the calculus of residue to evaluate $\int_{-\infty}^{\infty} \frac{cosx}{(x^2+a^2)(x^2+b^2)} dx$, (a > b > 0)

Solution.

Consider the integral $\int_C f(z)dz$ where f(z) = $\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

To find the poles of f(z):

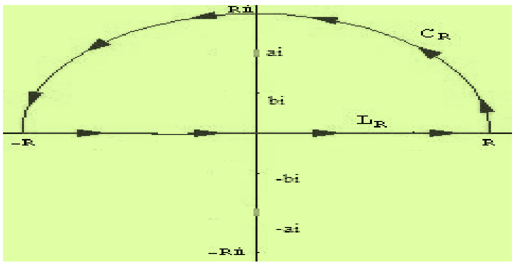
Poles of f(z) = zeros of $(z^2+a^2)(z^2+b^2)$, these zeros are given by $(z^2+a^2)(z^2+b^2) = 0$

$\Rightarrow z^2 = -a^2$ or $z^2 = -b^2 \Rightarrow z = (ai)^2$ or $z = (bi)^2 \Rightarrow z = \pm ai$ or $z = \pm bi$

⇒ z = ai or z = -ai or z =bi or z = -bi

Poles of f (z) are ai,-ai, bi,-bi (all are simple poles)

There are no poles on the real axis.



The poles lie inside the upper half of semi-circle are z = ai , z = bi

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ (on the real line –R to R (L_R)+ the upper half of the semi-circle C_R).

To find the residue of f(z):

Res[f,ai] = $\lim_{z \to ai}(z-ai) f(z) = \lim_{z \to ai}(z-ai) \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

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$= \lim_{z \to ai} (z - ai) \frac{e^{iz}}{(z+ai)(z-ai)(z+bi)(z-bi)} = \lim_{z \to ai} \frac{e^{iz}}{(z+ai)(z+bi)(z-bi)}$

$= \frac{e^{iai}}{(ai+ai)(ai+bi)(ai-bi)} = -\frac{e^{-a}}{2ai(a^2-b^2)} = \frac{e^{-a}}{2ai(b^2-a^2)}$

Similarly,

Res[f,bi] = $\lim_{z \to bi} (z - bi)\ f(z) = \frac{e^{-b}}{2bi(a^2-b^2)}$

Now $\lim_{z \to \infty} \frac{1}{(z^2+a^2)(z^2+b^2)} = 0$

By Jordan's Lemma,

$\lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} dz = 0$

As R $\to \infty$,

$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + \lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)} dx + 0$

$= \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)} dx$

Hence $\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)} dx = \int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k] = 2\pi i \left[\frac{e^{-a}}{2ai(b^2-a^2)} + \frac{e^{-b}}{2bi(a^2-b^2)}\right]$

$= \frac{2\pi i(be^{-a}-ae^{-b})}{2abi(b^2-a^2)} = \frac{\pi(be^{-a}-ae^{-b})}{ab(b^2-a^2)}$

$\Rightarrow \int_{-\infty}^{\infty} \frac{cosx+isinx}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi(be^{-a}-ae^{-b})}{ab(b^2-a^2)}$

Equating real and imaginary parts,

$\int_{-\infty}^{\infty} \frac{cosx}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi(be^{-a}-ae^{-b})}{ab(b^2-a^2)}$ and $\int_{-\infty}^{\infty} \frac{sinx}{(x^2+a^2)(x^2+b^2)} dx = 0$

Example III.3

Evaluate $\int_{-\infty}^{\infty} \frac{xcosx}{x^2+4} dx$ and $\int_{-\infty}^{\infty} \frac{xsinx}{x^2+4} dx$

Solution

Consider the integral $\int_C f(z)dz$ where f(z) = $\frac{ze^{iz}}{z^2+4}$

To find the poles of f(z):

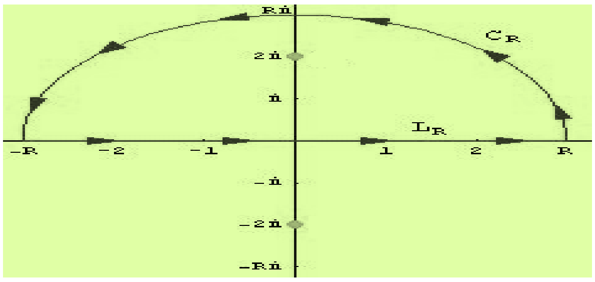
Poles of f(z) = zeros of (z^2+4), these zeros are given by $(z^2+ 4) =0$

$\Rightarrow z^2 = - 2^2 \Rightarrow z = (2i)^2 \Rightarrow z = \pm 2i$

$\Rightarrow$ z = 2i or z = -2i

Poles of f (z) are 2i,-2i (both simple poles)

There are no poles on the real axis.



The only pole lie inside the upper half of semi-circle is z = 2i

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

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Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ (on the real line –R to R (L_R)+ the upper half of the semi-circle C_R).

To find the residue of f(z):

Res[f,2i] = $\lim_{z\to 2i}(z-2i)\, f(z) = \lim_{z\to 2i}(z-2i)\,\frac{ze^{iz}}{z^2+4} = \lim_{z\to 2i}(z-2i)\,\frac{ze^{iz}}{(z+2i)(z-2i)}$

$= \lim_{z\to 2i}\frac{ze^{iz}}{(z+2i)} = \frac{2ie^{i2i}}{(2i+2i)} = \frac{e^{-2}}{2}$

Now $\lim_{z\to\infty}\frac{z}{(z^2+4)} = 0$

By Jordan's Lemma,

$\lim_{R\to\infty}\int_{C_R}\frac{ze^{iz}}{(z^2+4)}dz = 0$

As R → ∞ ,

$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + \lim_{R\to\infty}\int_{C_R}\frac{ze^{iz}}{(z^2+4)}dz = \int_{-\infty}^{\infty}\frac{xe^{ix}}{(x^2+4)}dx + 0$

Hence $\int_{-\infty}^{\infty}\frac{xe^{ix}}{(x^2+4)}dx = \int_C f(z)dz = 2\pi i\sum \text{Res}[f,z_k] = 2\pi i\left[\frac{e^{-2}}{2}\right] = e^{-2}\pi i$

$\Rightarrow \int_{-\infty}^{\infty}\frac{x\cos x+ix\sin x}{(x^2+4)}dx = e^{-2}\pi i$

Equating real and imaginary parts,

$\int_{-\infty}^{\infty}\frac{x\cos x}{(x^2+4)}dx = 0$ and $\int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)}dx = e^{-2}\pi$

Example III.4

Evaluate $\int_0^{\infty}\frac{\cos mx}{x^4+x^2+1}dx$ (m > 0)

Solution

Consider the integral $\int_C f(z)dz$ where f(z) = $\frac{e^{imz}}{z^4+z^2+1}$

To find the poles of f(z):

Poles of f(z) = zeros of (z^4+z^2+1), these zeros are given by $z^4+z^2+1=0$

$z^4+- z^2 + z^2+ z^2 +1=0 \Rightarrow z^2 (z^2 +1) + z^2 +1 = z^2$

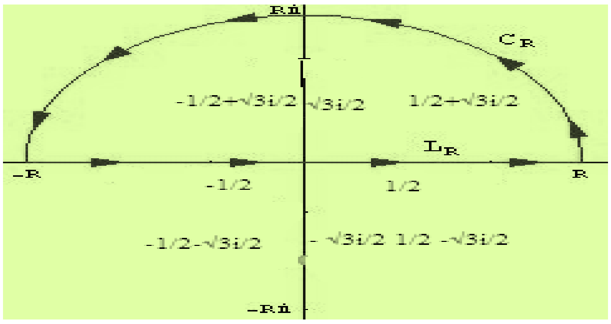
$\Rightarrow (z^2 +1)^2 - z^2 = 0 \Rightarrow (z^2 +1-z)(z^2 +1+z) = 0$

$\Rightarrow z^2 +1-z = 0$ or $z^2 +1+z = 0$

$\Rightarrow z = \frac{1+\sqrt{3}i}{2}$,or $z = \frac{1-\sqrt{3}i}{2}$, or $z = \frac{-1+\sqrt{3}i}{2}$,or $z = \frac{-1-\sqrt{3}i}{2}$

Poles of f (z) are $\frac{1+\sqrt{3}i}{2}$, $\frac{1-\sqrt{3}i}{2}$, $\frac{-1+\sqrt{3}i}{2}$, $\frac{-1-\sqrt{3}i}{2}$ (all are simple poles)

There are no poles on the real axis.



The poles lie inside the upper half of semi-circle is z = $\frac{1+\sqrt{3}i}{2}$ = α (say) and z = $\frac{-1+\sqrt{3}i}{2}$ = β(say)

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i\sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

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Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ (on the real line –R to R (L_R) + the upper half of the semi-circle C_R).

To find the residue of f(z):

$$\text{Res}[f,\alpha] = \lim_{z\to\alpha}(z-\alpha)\, f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{e^{imz}}{(z-\alpha)(z-\beta)(z+\alpha)(z+\beta)} = \lim_{z\to\alpha}\frac{e^{imz}}{(z-\beta)(z+\alpha)(z+\beta)}$$

$$= \frac{e^{im\alpha}}{(\alpha-\beta)(\alpha+\alpha)(\alpha+\beta)} = \frac{e^{im\alpha}}{(\alpha-\beta)(2\alpha)(\alpha+\beta)} = \frac{e^{im\alpha}}{(\alpha-\beta)(2\alpha)(\alpha+\beta)} = \frac{e^{\frac{im(1+\sqrt{3}i)}{2}}}{[\frac{1+\sqrt{3}i}{2}-\frac{-1+\sqrt{3}i}{2}](\frac{2(1+\sqrt{3}i)}{2})[\frac{(1+\sqrt{3}i)}{2}+\frac{(-1+\sqrt{3}i)}{2}]}$$

$$= \frac{e^{\frac{im(1+\sqrt{3}i)}{2}}}{(\frac{2(1+\sqrt{3}i)}{2})\left(\frac{2\sqrt{3}i}{2}\right)} = \frac{e^{\frac{im(1+\sqrt{3}i)}{2}}}{(\sqrt{3}i-3)} = \frac{e^{\frac{im}{2}}e^{\frac{-\sqrt{3}m}{2}}}{(\sqrt{3}i-3)}$$

Similary,

$$\text{Res}[f,\beta] = \frac{e^{\frac{im(-1+\sqrt{3}i)}{2}}}{[\frac{(-1+\sqrt{3}i)}{2}-\frac{1+\sqrt{3}i}{2}](\frac{2(-1+\sqrt{3}i)}{2})[\frac{(1+\sqrt{3}i)}{2}+\frac{(-1+\sqrt{3}i)}{2}]} = \frac{e^{\frac{im(-1+\sqrt{3}i)}{2}}}{\frac{(-1)\left(2(-1+\sqrt{3}i)\right)}{2}(2\sqrt{3}i)}$$

$$\frac{e^{\frac{im(-1+\sqrt{3}i)}{2}}}{(-1)(\frac{2(-1+\sqrt{3}i)}{2})\left(\frac{2\sqrt{3}i}{2}\right)} = \frac{e^{\frac{im(-1+\sqrt{3}i)}{2}}}{(\sqrt{3}i+3)} = \frac{e^{-\frac{im}{2}}e^{\frac{-\sqrt{3}m}{2}}}{(\sqrt{3}i+3)}$$

Now $\lim_{z\to\infty} \frac{1}{(z^4+z^2+1)} = 0$

By Jordan's Lemma,

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{imz}}{(z^4+z^2+1)}dz = 0$$

As R → ∞ ,

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + \lim_{R\to\infty}\int_{C_R}\frac{e^{imz}}{(z^4+z^2+1)}dz = \int_{-\infty}^{\infty}\frac{e^{imx}}{(x^4+x^2+1)}dx + 0$$

Hence $\int_{-\infty}^{\infty}\frac{e^{imx}}{(x^4+x^2+1)}dx = \int_C f(z)dz = 2\pi i\sum \text{Res}[f,z_k] = 2\pi i\left[\frac{e^{\frac{im}{2}}e^{\frac{-\sqrt{3}m}{2}}}{(\sqrt{3}i-3)} + \frac{e^{-\frac{im}{2}}e^{\frac{-\sqrt{3}m}{2}}}{(\sqrt{3}i+3)}\right]$

$$= 2\pi i\left[\frac{e^{\frac{-\sqrt{3}m}{2}}\left\{(3+\sqrt{3}i)e^{\frac{im}{2}}+(\sqrt{3}i-3)e^{-\frac{im}{2}}\right\}}{(\sqrt{3}i-3)(\sqrt{3}i+3)}\right]$$

$$= 2\pi i\left[\frac{e^{\frac{-\sqrt{3}m}{2}}}{}\frac{\left\{(3+\sqrt{3}i)\left(\cos\left(\frac{m}{2}\right)+i\sin\left(\frac{m}{2}\right)\right)+(\sqrt{3}i-3)\left(\cos\left(\frac{m}{2}\right)-i\sin\left(\frac{m}{2}\right)\right)\right\}}{(-3-9)}\right]$$

$$= 2\pi i\left[\frac{e^{\frac{-\sqrt{3}m}{2}}\left\{(3+\sqrt{3}i)\left(\cos\left(\frac{m}{2}\right)+i\sin\left(\frac{m}{2}\right)\right)+(\sqrt{3}i-3)\left(\cos\left(\frac{m}{2}\right)-i\sin\left(\frac{m}{2}\right)\right)\right\}}{-12}\right]$$

$$= 2\pi i\left[\frac{e^{\frac{-\sqrt{3}m}{2}}\left\{2\sqrt{3}i\cos\left(\frac{m}{2}\right)+6i\sin\left(\frac{m}{2}\right)\right\}}{-12}\right]$$

$$= \frac{-4\pi e^{\frac{-\sqrt{3}m}{2}}(\sqrt{3}\cos\left(\frac{m}{2}\right)+3\sin\left(\frac{m}{2}\right))}{-12} = \frac{\pi e^{\frac{-\sqrt{3}m}{2}}(\sqrt{3}\cos\left(\frac{m}{2}\right)+3\sin\left(\frac{m}{2}\right))}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos mx+i\sin mx}{(x^4+x^2+1)}dx = \frac{\pi e^{\frac{-\sqrt{3}m}{2}}(\sqrt{3}\cos\left(\frac{m}{2}\right)+3\sin\left(\frac{m}{2}\right))}{3}$$

Equating real and imaginary parts,

$$\int_{-\infty}^{\infty}\frac{\cos mx}{(x^4+x^2+1)}dx = \frac{\pi e^{\frac{-\sqrt{3}m}{2}}(\sqrt{3}\cos\left(\frac{m}{2}\right)+3\sin\left(\frac{m}{2}\right))}{3}$$

and $\int_{-\infty}^{\infty} \frac{sinmx}{(x^4+x^2+1)} dx = 0$

Hence $\int_{0}^{\infty} \frac{cosmx}{(x^4+x^2+1)} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{cosmx}{(x^4+x^2+1)} dx = \frac{\pi e^{\frac{-\sqrt{3}m}{2}}(\sqrt{3}\cos(\frac{m}{2}) + 3sin(\frac{m}{2}))}{6}$

Example III.5

Prove that $\int_{0}^{\infty} \frac{cosmx}{(a^2+x^2)^2} dx = \frac{\pi}{2a^3}(1+ma)e^{-ma}$ (m > 0, a > 0)

Solution

Consider the integral $\int_C f(z)dz$ where f(z) = $\frac{e^{imz}}{(a^2+z^2)^2}$

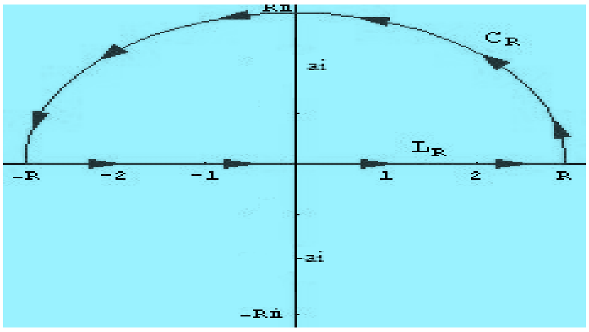
To find the poles of f(z):

Poles of f(z) = zeros of $(z^2+a^2)^2$, these zeros are given by $(z^2+a^2)^2=0$

$\Rightarrow z^2 = -a^2 \Rightarrow z = (ai)^2 \Rightarrow z = \pm ai \Rightarrow z = ai$ (twice)or z = -ai(twice)

Poles of f (z) are ai, -ai (both are order 2)

There are no poles on the real axis



The only pole lie inside the upper half of semi-circle is z = ai (order 2)

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum Res[f,z_k]$ where z_k are the singularities(poles) of f(z).

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ (on the real line –R to R (L_R)+ the upper half of the semi-circle C_R).

To find the residue of f(z):

$$Res[f,ai] = \lim_{z \to ai} \frac{d}{dz}(z-ai)^2 f(z) = \lim_{z \to ai} \frac{d}{dz}(z-ai)^2 \frac{e^{imz}}{(a^2+z^2)^2}$$

$$= \lim_{z \to ai} \frac{d}{dz}(z-ai)^2 \frac{e^{imz}}{(z+ai)^2(z-ai)^2} = \lim_{z \to ai} \frac{d}{dz} \frac{e^{imz}}{(z+ai)^2} = \lim_{z \to ai} \frac{[(z+ai)^2 ime^{imz} - e^{imz}2(z+ai)]}{(z+ai)^4}$$

$$= \lim_{z \to ai} \frac{[(z+ai)ime^{imz} - e^{imz}2]}{(z+ai)^3} = \frac{[(ai+ai)ime^{imai} - e^{imai}2]}{(ai+ai)^3} = \frac{[-2ame^{-ma} - 2e^{-ma}]}{(2ai)^3} = -\frac{2e^{-ma}(ma+1)}{-8a^3 i}$$

$$= \frac{e^{-ma}(ma+1)}{4a^3 i}$$

Now $\lim_{z \to \infty} \frac{1}{(z^2+a^2)^2} = 0$

$\therefore$ by Jordan's lemma , $\lim_{R \to \infty} \int_{C_R} \frac{e^{imz}}{(z^2+a^2)^2} dz = 0$

As $R \to \infty$,

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + \lim_{R \to \infty} \int_{C_R} \frac{e^{imz}}{(z^2+a^2)^2} dz = \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2} dx + 0$$

$$= \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2} dx$$

Hence $\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2} dx = \int_C f(z)dz = 2\pi i \sum Res[f,z_k] = 2\pi i \frac{e^{-ma}(ma+1)}{4a^3 i} = \frac{\pi e^{-ma}(ma+1)}{2a^3}$

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$\Rightarrow \int_{-\infty}^{\infty} \frac{cosmx+isinmx}{(x^2+a^2)^2} dx = \frac{\pi e^{-ma}(ma+1)}{2a^3}$

$\Rightarrow \int_{-\infty}^{\infty} \frac{cosmx}{(x^2+a^2)^2} dx + i\int_{-\infty}^{\infty} \frac{sinmx}{(x^2+a^2)^2} dx = \frac{\pi e^{-ma}(ma+1)}{2a^3}$

Equating real and imaginary parts,

$\int_{-\infty}^{\infty} \frac{cosmx}{(x^2+a^2)^2} dx = \frac{\pi e^{-ma}(ma+1)}{2a^3}$ and $\int_{-\infty}^{\infty} \frac{sinmx}{(x^2+a^2)^2} dx = 0$

$\Rightarrow \int_{0}^{\infty} \frac{cosmx}{(x^2+a^2)^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{cosmx}{(x^2+a^2)^2} dx = \frac{\pi e^{-ma}(ma+1)}{4a^3}$

Note:III.1

$z = re^{i\theta} \Rightarrow r = |z|$ and $\theta = \arg(z)$

$\log z = \text{Log } r + i\arg(z)$

If $z = x+iy$, $r = (x^2+y^2)^{1/2}$, $\theta = \arg(z) = \arg(x+iy) = \tan^{-1}(y/x)$

$\log(x+i) = \log(x^2+1)^{1/2} + i\arg(x) = \log(x^2+1)^{1/2} + 0 = \log(x^2+1)^{1/2}$

Example III.6

Prove that $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi log2$

Solution

Consider the integral $\int_C f(z)dz$ where $f(z) = \frac{\log(z+i)}{z^2+1}$

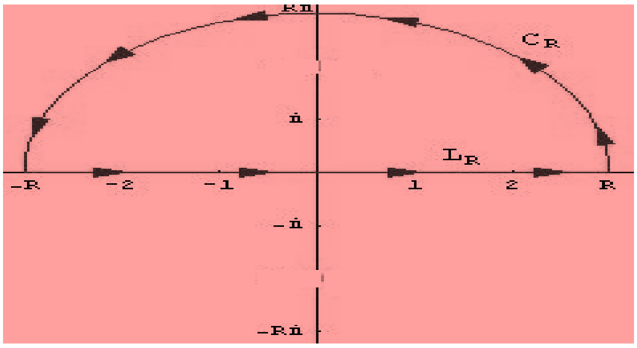
To find the poles of f(z):

Poles of f(z) = zeros of (z^2+1), these zeros are given by $(z^2+1)=0$

$\Rightarrow z^2 = -1 \Rightarrow z = (i)^2 \Rightarrow z = \pm i \Rightarrow z = i$ or $z = -i$

Poles of f (z) are i, -i (both are simple poles)

There are no poles on the real axis.



The only pole lie inside the upper half of semi-circle is z = i

By Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k]$ where z_k are the singularities(poles) of f(z).

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$ (on the real line –R to R (L_R)+ the upper half of the semi-circle C_R).

To find the residue of f(z):

$\text{Res}[f,i] = \lim_{z \to i}(z-i)\, f(z) = \lim_{z \to i}(z-i)\frac{\log(z+i)}{z^2+1} = \lim_{z \to i}(z-i)\frac{\log(z+i)}{(z-i)(z+i)}$

$= \lim_{z \to i}\frac{\log(z+i)}{(z+i)} = \frac{\log(i+i)}{(i+i)} = \frac{\log 2i}{2i} = \frac{\log(2^2)^{\frac{1}{2}}+i\tan^{-1}(\frac{2}{0})}{2i} = \frac{\log 2+i\tan^{-1}(\infty)}{2i}$ (using Note III.1)

$= \frac{\log 2+\frac{i\pi}{2}}{2i}$

Now $\lim_{z \to \infty}\frac{log(z+i)}{z^2+1} = \lim_{z \to \infty}\frac{log(z+i)}{(z+i)(z-i)} = \lim_{z \to \infty}\frac{1}{(z-i)}\lim_{z \to \infty}\frac{log(z+i)}{(z+i)}$

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Consider $\lim\limits_{z \to \infty} \frac{1}{(z-i)} = 0$

Consider $\lim\limits_{z \to \infty} \frac{\log(z+i)}{(z+i)}$ it is undetermined, so we have to use L'Hospital's rule

$\lim\limits_{z \to \infty} \frac{\frac{1}{z+i}}{1} = \lim\limits_{z \to \infty} \frac{1}{z+i} = 0$

Hence $\lim\limits_{z \to \infty} \frac{\log(z+i)}{z^2+1} = 0$

$\Rightarrow \lim\limits_{z \to \infty} \int_{C_R} \frac{\log(z+i)}{z^2+1} = 0 \Rightarrow \lim\limits_{R \to \infty} \int_{C_R} \frac{\log(z+i)}{z^2+1} = 0$ ($\because |z| = R$)

As $R \to \infty$,

$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + \lim\limits_{R \to \infty} \int_{C_R} \frac{\log(z+i)}{z^2+1} dz = \int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1} dx + 0$

$= \int_{-\infty}^{\infty} \frac{\log(x^2+1)^{\frac{1}{2}}}{x^2+1} dx = \int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(x^2+1)}{x^2+1} dx$ (By using the Note III.1)

$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1} dx = \int_0^{\infty} \frac{\log(x^2+1)}{x^2+1} dx$

Hence $\int_0^{\infty} \frac{\log(x^2+1)}{x^2+1} dx = \int_C f(z)dz = 2\pi i \sum \text{Res}[f,z_k] = 2\pi i \frac{\log 2 + \frac{i\pi}{2}}{2i}$

$= \pi\log 2 + i\frac{\pi^2}{2}$

Equating real part $\int_0^{\infty} \frac{\log(x^2+1)}{x^2+1} dx = \pi\log 2$

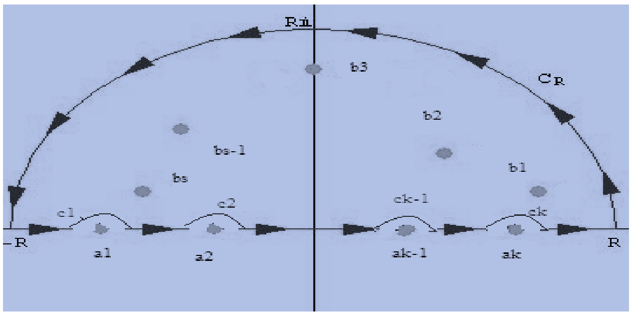
Case of poles are on the real axis.

Type IV

If the rational function $f(z) = \frac{g(z)}{h(z)}$, then degree of $h(z)$ exceeds that of $g(z)$ and $g(z) \neq 0$.

Suppose h(z) has simple zeros on the real axis (that is simple poles of f(z) on the real axis) , let it be $a_1, a_2, \ldots a_k$ and h(z) has zeros inside the upper half of semi-circle (that is poles of f(z) inside the upper half of semi-circle), let it be $b_1, b_2, \ldots b_s$,

then $\int_{-\infty}^{\infty} f(x)dx = \pi i \sum \text{Res}[f,a_k] + 2\pi i \sum \text{Res}[f,b_s]$, where k = 1,2,….k and s = 1,2,…s



Where $C_1, C_2, \ldots C_k$ are the semi circles and $b_1, b_2, \ldots b_s$ are lie upper half of these semi circles.

Example IV.1.

Evaluate $\int_{-\infty}^{\infty} \frac{x}{x^3-8} dx$

Solution.

Consider the integral $\int_C f(z)dz$ where $f(z) = \frac{z}{z^3-8}$

To find the poles of f(z):

Poles of f(z) = zeros of ($z^3 - 8$), these zeros are given by ($z^3 - 8$)=0

$\Rightarrow z^3 = 8 \Rightarrow z^3 = (2)^3 \Rightarrow z = 2$

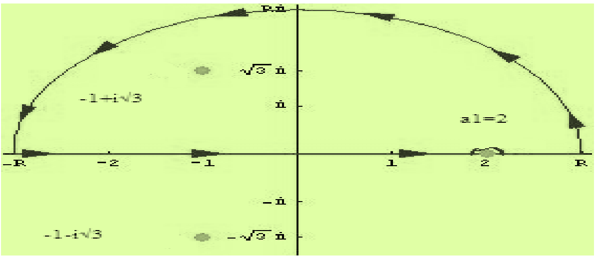
Since z-2 is a factor of $z^3 - 8$, $z^3 - 8 = (z-2)(z^2+2z+4) = 0$

$\Rightarrow z^2+2z+4 = 0 \Rightarrow z = \frac{-2\pm\sqrt{4-16}}{2} = \frac{-2\pm\sqrt{-4*3}}{2} = \frac{-2\pm 2i\sqrt{3}}{2} = -1\pm i\sqrt{3}$

Poles of f (z) are 2 , $-1+i\sqrt{3}$ and $-1-i\sqrt{3}$ (all are simple poles)

Pole lie on the real axis z =2

Pole lie inside the upper half of semi-circle $z = -1+i\sqrt{3}$



To find the residue of f(z):

$\text{Res}[f,2] = \lim_{z \to 2}(z-2)\ f(z) = \lim_{z \to 2}(z-2)\ \frac{z}{z^3-8} = \lim_{z \to 2}(z-2)\ \frac{z}{(z-2)(z+1-i\sqrt{3})(z+1+i\sqrt{3})}$

$= \lim_{z \to 2} \frac{z}{(z+1-i\sqrt{3})(z+1+i\sqrt{3})} = \frac{2}{(2+1-i\sqrt{3})(2+1+i\sqrt{3})} = \frac{2}{(3-i\sqrt{3})(3+i\sqrt{3})}$

$= \frac{2}{(9+3)} = \frac{1}{6}$

$\text{Res}[f, -1+i\sqrt{3}] = \lim_{z \to -1+i\sqrt{3}} (z-(-1+i\sqrt{3}))\ f(z)$

$= \lim_{z \to -1+i\sqrt{3}} (z+1-i\sqrt{3})\ \frac{z}{z^3-8} = \lim_{z \to -1+i\sqrt{3}} (z+1-i\sqrt{3})\ \frac{z}{(z-2)(z+1-i\sqrt{3})(z+1+i\sqrt{3})}$

$= \lim_{z \to -1+i\sqrt{3}} \frac{z}{(z-2)(z+1+i\sqrt{3})} = \frac{-1+i\sqrt{3}}{(-1+i\sqrt{3}-2)(-1+i\sqrt{3}+1+i\sqrt{3})} = \frac{-1+i\sqrt{3}}{(-3+i\sqrt{3})(2i\sqrt{3})} = \frac{-1+i\sqrt{3}}{(-6i\sqrt{3}-6)}$

$= \frac{-1+i\sqrt{3}}{-6(i\sqrt{3}+1)} = \frac{(-1+i\sqrt{3})(1-i\sqrt{3})}{-6(i\sqrt{3}+1)(1-i\sqrt{3})} = \frac{-(-1+i\sqrt{3})(-1+i\sqrt{3})}{-6(1+3)} = \frac{(1-3-2i\sqrt{3})}{24} = \frac{(-2-2i\sqrt{3})}{24} = \frac{-2(1+i\sqrt{3})}{24} = \frac{-(1+i\sqrt{3})}{12}$

We know that $\int_{-\infty}^{\infty} f(x)dx = \pi i \sum \text{Res}[f,a_k] + 2\pi i \sum \text{Res}[f,b_s]$ where a_k 's are the poles lie on real axis and b_s 's are the poles lie inside the upper half of semi-circle.

$\int_{-\infty}^{\infty} \frac{x}{x^3-8} dx = \pi i \sum \text{Res}[f,a_k] + 2\pi i \sum \text{Res}[f,b_s] = \pi i(\frac{1}{6}) + 2\pi i(\frac{-(1+i\sqrt{3})}{12})$

$= \frac{\pi i}{6} - \frac{\pi i}{6} - \frac{\pi i\sqrt{3}i}{6} = \frac{\sqrt{3}\pi}{6}$

Type V

If the rational function $f(z) = \frac{g(z)}{h(z)}$, then degree of h(z) exceeds that of g(z) and $g(z) \neq 0$.

Suppose h(z) has simple zeros on the real axis (that is simple poles of f(z) on the real axis) , let it be $a_1, a_{2,\dots} a_k$

and h(z) has zeros inside the upper half of semi-circle (that is poles of f(z) inside the upper half of semi-circle), let it be $b_1, b_{2,\dots} b_{s,}$

Let m be a positive real number and if $f(z) = \frac{e^{imz} g(z)}{h(z)}$, then

$\int_{-\infty}^{\infty} cosmx \frac{g(x)}{h(x)} dx \quad = \text{Re} \int_{-\infty}^{\infty} cosmx\ f(x)dx$

$= Re\left[2\pi i \sum_{i=1}^{s} \text{Res}[f, bi]\right] + Re\left[\pi i \sum_{j=1}^{k} \text{Res}[f, aj]\right]$

And

$\int_{-\infty}^{\infty} sinmx \frac{g(x)}{h(x)} dx \quad = \text{Img} \int_{-\infty}^{\infty} sinmx\ f(x)dx$

$= Img\left[2\pi i \sum_{i=1}^{s} \text{Res}[f, bi]\right] + Img\left[\pi i \sum_{j=1}^{k} \text{Res}[f, aj]\right]$

Where $b_1, b_{2,\dots} b_{s,}$ are the poles of f(z) that lie in the upper half of the semi-circles $C_1, C_2, \dots C_k$.

Example V.1.

Prove that $\int_{-\infty}^{\infty} \frac{\cos x}{(x-1)(x^2+4)} dx = \frac{\pi}{10}\left(-\frac{1}{e^2} - 2\sin 1\right)$ and

$\int_{-\infty}^{\infty} \frac{\sin x}{(x-1)(x^2+4)} dx = \frac{\pi}{5}\left(-\frac{1}{e^2} + \cos 1\right)$

Solution.

Consider the integral $\int_C f(z)dz$ where $f(z) = \frac{e^{iz}}{(z-1)(z^2+4)}$

To find the poles of f(z):

Poles of f(z) = zeros of $(z-1)(z^2+4)$, these zeros are given by $(z-1)(z^2+4) = 0$

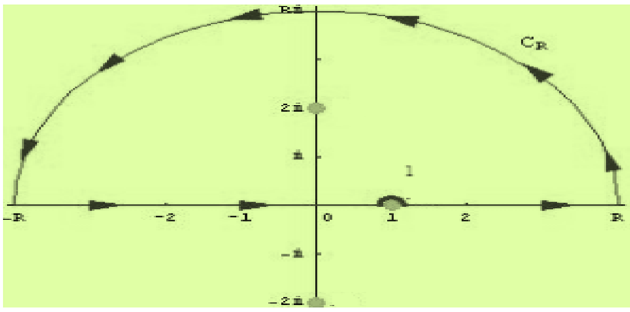
$\Rightarrow z-1 = 0$ or $z^2+4 = 0 \Rightarrow z = 1$ or $z^2 = -4 = (2i)^2$

$\Rightarrow z = 1$ or $z = 2i$ or $z = -2i$

Poles of f(z) are z = 1 , z = 2i , z = -2i (all are simple poles)

The only pole lie on the real axis is z = 1

The only pole lie inside the semi-circle is z = 2i



To find the residue of f(z):

$\text{Res}[f,1] = \lim_{z \to 1} (z-1)\, f(z) = \lim_{z \to 1} (z-1) \frac{e^{iz}}{(z-1)(z^2+4)} = \lim_{z \to 1} \frac{e^{iz}}{(z^2+4)}$

$= \frac{e^i}{(1+4)} = \frac{e^i}{5}$

$\text{Res}[f,2i] = \lim_{z \to 2i} (z-2i)\, f(z) = \lim_{z \to 2i} (z-2i) \frac{e^{iz}}{(z-1)(z+2i)(z-2i)}$

$= \lim_{z \to 2i} \frac{e^{iz}}{(z-1)(z+2i)} = \frac{e^{i2i}}{(2i-1)(2i+2i)} = \frac{e^{-2}}{(2i-1)(4i)}$

We know that

$\int_{-\infty}^{\infty} \cos mx \frac{g(x)}{h(x)} dx =$

$Re \int_{-\infty}^{\infty} \cos mx\, f(x)dx = Re\left[2\pi i \sum_{i=1}^{s} \text{Res}[f, bi]\right] + Re\left[\pi i \sum_{j=1}^{k} \text{Res}[f, aj]\right]$

And $\int_{-\infty}^{\infty} \sin mx \frac{g(x)}{h(x)} dx =$

$Img \int_{-\infty}^{\infty} \sin mx\, f(x)dx = Img\left[2\pi i \sum_{i=1}^{s} \text{Res}[f, bi]\right] + Img\left[\pi i \sum_{j=1}^{k} \text{Res}[f, aj]\right]$

Where a_k's are the poles lie on the real axis and b_s's are the poles lie inside the upper half of the semi-circle.

$\int_{-\infty}^{\infty} \frac{\cos x}{(x-1)(x^2+4)} dx = \text{Re} \int_{-\infty}^{\infty} \cos x\, f(x)dx = \text{Re} \int_{-\infty}^{\infty} \cos x \frac{e^{ix}}{(x-1)(x^2+4)} dx = Re\left[2\pi i \text{ Res}[f, 2i]\right] + Re\left[\pi i \text{ Res}[f, 1]\right]$

$= Re\left[2\pi i\, \frac{e^{-2}}{(2i-1)(4i)}\right] + Re\left[\pi i\, \frac{e^{i}}{5}\right]$

$= Re\left[\pi\, \frac{e^{-2}}{(2i-1)(2)}\right] + Re\left[\pi i\, (\frac{cos1+isin1}{5})\right]$

$= Re\left[\pi\, \frac{e^{-2}(2i+1)}{2(2i-1)(2i+1)}\right] + Re\left[\frac{\pi icos1-\pi sin1}{5}\right] = Re\left[\pi\, \frac{e^{-2}(2i+1)}{2(-4-1)}\right] - \frac{\pi sin1}{5} = Re\left[\pi\, \frac{e^{-2}(2i+1)}{-10}\right] - \frac{\pi sin1}{5}$

$= \left[\frac{\pi e^{-2}}{-10}\right] - \frac{\pi sin1}{5} = -\frac{\pi}{10}[\,e^{-2} + 2sin1]$

$\int_{-\infty}^{\infty} \frac{\sin x}{(x-1)(x^2+4)}dx = \text{Img} \int_{-\infty}^{\infty} sinx\, f(x)dx = \text{Img} \int_{-\infty}^{\infty} cosx\, \frac{e^{ix}}{(x-1)(x^2+4)}dx$

$= Img\left[2\pi i \sum_{i=1}^{s} \text{Res[f, bi]}\right] + Img\left[\pi i \sum_{j=1}^{k} \text{Res[f, aj]}\right]$

$= \text{Img}\left[\pi\, \frac{e^{-2}(2i+1)}{2(2i-1)(2i+1)}\right] + Img\left[\frac{\pi icos1-\pi sin1}{5}\right]$

$= Img\left[\pi\, \frac{e^{-2}(2i+1)}{-10}\right] + \frac{\pi cos1}{5} = \pi\, \frac{e^{-2}2}{-10} + \frac{\pi cos1}{5} = \pi\, \frac{e^{-2}}{-5} + \frac{\pi cos1}{5} = \frac{\pi}{5}[-e^{-2} + cos1]$

Example V.2.

Prove that $\int_{0}^{\infty} \frac{\sin mx}{x}dx = \frac{\pi}{2}$

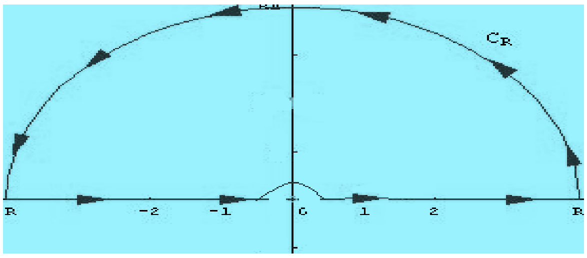
Solution.

Consider the integral $\int_C f(z)dz$ where $f(z) = \frac{e^{imz}}{z}$

To find the poles of f(z):

Poles of f(z) = zeros of (z), this zero is given by z =0

The only pole of f(z) is z = 0 simple and lie on real axis



To find the residue of f(z):

$\text{Res[f,0]} = \lim_{z \to 0}(z-0)\, f(z) = \lim_{z \to 0} z\, \frac{e^{izm}}{z} = e^0 = 1$

We know that $\int_{-\infty}^{\infty} cosmx \frac{g(x)}{h(x)}dx =$

$Re \int_{-\infty}^{\infty} cosmx\, f(x)dx = Re\left[2\pi i \sum_{i=1}^{s} \text{Res[f, bi]}\right] + Re\left[\pi i \sum_{j=1}^{k} \text{Res[f, aj]}\right]$

And $\int_{-\infty}^{\infty} Sinmx \frac{g(x)}{h(x)}dx =$

$Img \int_{-\infty}^{\infty} sinmx\, f(x)dx = Img\left[2\pi i \sum_{i=1}^{s} \text{Res[f, bi]}\right] + Img\left[\pi i \sum_{j=1}^{k} \text{Res[f, aj]}\right]$

Where a_k's are the poles lie on the real axis and b_s's are the poles lie inside the upper half of the semi-circle.

$\therefore \int_{-\infty}^{\infty} \frac{sinmx}{x}dx = Img \int_{-\infty}^{\infty} sinmx\, f(x)dx = Img \int_{-\infty}^{\infty} sinmx \frac{e^{imx}}{x}dx$

$= Img\left[2\pi i \sum_{i=1}^{s} \text{Res[f, bi]}\right] + Img\left[\pi i \sum_{j=1}^{k} \text{Res[f, aj]}\right]$

$= Img\left[\pi i\, \text{Res[f, 0]}\right] = Img[\pi i\,(1)] = \pi$

Hence $\int_{0}^{\infty} \frac{sinmx}{x}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{sinmx}{x}dx = \frac{\pi}{2}$

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5. CONCLUSION

In careful consideration and comparisons of various results in contour integration made an important development in finding more concepts in engineering fields and I am still working on this to get more results in these areas. The extension of the contour integral method is used for the electrical design of planar structures in digital systems and this method is further developed for analysis of multi-media circuits and various engineering files.

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